Classical notions of computation and the Hasegawa-Thielecke theorem (extended version)

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In the spirit of the Curry-Howard correspondence between proofs and programs, we define and study a syntax and semantics for classical logic equipped with a computationally involutive negation, using a polarised effect calculus, the linear classical L-calculus. A main challenge in designing a denotational semantics for the calculus is to accommodate both call-by-value and call-by-name evaluation strategies, which leads to a failure of associativity of composition. In order to tackle this issue, we define a notion of adjunction between graph morphisms on non-associative categories, which we use to formulate polarized and non-associative notions of *symmetric monoidal closed duploid* and of *dialogue duploid*. We show that they provide a direct style counterpart to adjunction models: *linear effect adjunctions* for the (linear) call-by-push-value calculus and *dialogue chiralities* for linear continuations, respectively. In particular, we show that the syntax of the linear classical L-calculus can be interpreted in any dialogue duploid, and that it defines in fact a syntactic dialogue duploid. As an application, we establish, by semantic as well as syntactic means, the Hasegawa-Thielecke theorem, which states that the notions of central map and of thunkable map coincide in any dialogue duploid (in particular, for any double negation monad on a symmetric monoidal category).

CCS Concepts: • **Theory of computation** → *Proof theory*; *Categorical semantics*; *Linear logic*; *Lambda calculus*.

Additional Key Words and Phrases: Classical logic, Continuations, Effects, Call-by-push-value

1 Introduction

1.1 Emergence of non-associativity between call-by-value and call-by-name

In this paper, we combine methods from proof theory and programming language semantics to investigate the meaning of expressions for proofs or programs, starting with those of the form

$$\textbf{let } a = u \textbf{ in } t \tag{1}$$

where t and u are effectful expressions, and where t possibly contains free instances of the variable a One main difficulty we face is that there are two canonical ways of assigning meaning to the **let** construct (1) depending on the evaluation paradigm at work:

In the call-by-value (CBV) paradigm, the expression u performs a number of actions and returns a value v ; the value v is then substituted for every free instance of the variable a in the expression t ; it is then the turn of the expression $t[a := v]$ to perform its actions and to return a value.

In the call-by-name (CBN) paradigm, the expression t performs its actions and is evaluated while the expression u is "frozen" and substituted for each free instance / of the variable a in t ; a new copy of the expression u performs its actions and is evaluated each time a free instance of the variable a appears as head variable during the evaluation of the expression $t[a := u]$.

By way of illustration, consider the probabilistic program

$$\textbf{let } x = (1/3 \, \textbf{true} + 2/3 \, \textbf{false}) \textbf{ in } t$$

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which returns a probabilistic distribution of values of a given type computed by the pure program t defined as

$$\begin{array}{rl} \mathbf{if}\ x = \mathbf{true}\ \ \mathbf{then} & (\mathbf{if}\ x = \mathbf{true}\ \mathbf{then}\ a\ \mathbf{else}\ b) \\ \mathbf{else} & (\mathbf{if}\ x = \mathbf{true}\ \mathbf{then}\ c\ \mathbf{else}\ d) \end{array}$$

where a, b, c, d are values of the given type. The meaning of the probabilistic program is equal to the distribution

$$\begin{array}{lcll} u & = & 1/3\, a + 2/3\, d & \text{in call by value, and} \\ u & = & 1/9\, a + 2/9\, b + 2/9\, c + 4/9\, d & \text{in call by name.} \end{array}$$

Note that the value of a is sampled once and for all during the call-by-value evaluation, while it is sampled twice during the call-by-name evaluation. This explains that the summands b and c do not appear in the former case, and that fractions over $9 = 3 \times 3$ appear in the latter case.

Kleisli categories. The seminal work on computational effects by Moggi [1989, 1991] initiated a well-established tradition of interpreting CBV expressions of type $a : A \vdash t : B$ as maps $t : A \to B$ in the Kleisli category $\mathbf{Kl}[\mathscr{C}, T]$ associated to a monad (T, μ, η) on a category $\mathscr{C}$ ([Filinski 1994; Power and Robinson 1997; Führmann 1999; Levy 1999; Plotkin and Power 2002; Power 2002; Führmann and Thielecke 2004; Lindley and Stark 2005; Katsumata 2005] among others). Recall that a map $f : A \to B$ in the Kleisli category is a map $f : A \to TB$ in the original category $\mathscr{C}$ and that two maps $f : A \to TB$ and $g : B \to TC$ are composed using the multiplication μ of the monad:

$$g \bullet f \quad = \quad A \xrightarrow{f} TB \xrightarrow{tg} TTC \xrightarrow{\mu_C} TC$$

Symmetrically, there is a well-established tradition after Girard [1987] of interpreting CBN expressions of type $a : A \vdash t : B$ as maps $t : A \to B$ in the co-Kleisli category $\mathbf{coKl}[\mathscr{C}, K]$ associated to a computational comonad (K, δ, ε) on a given category $\mathscr{C}$ of types and pure programs. Recall that a map $f : A \to B$ in the co-Kleisli category is a map $f : KA \to B$ in the original category $\mathscr{C}$ and that two maps $f : A \to B$ and $g : B \to C$ are composed in the co-Kleisli category using the comultiplication δ of the comonad:

$$g \circ f \quad = \quad KA \xrightarrow{\delta_A} KKA \xrightarrow{Kf} KB \xrightarrow{g} C$$

The mathematical property that composition is *associative* in $\mathbf{Kl}[\mathscr{C}, T]$ and $\mathbf{coKl}[\mathscr{C}, K]$, that is:

$$h \bullet (g \bullet f) \ = \ (h \bullet g) \bullet f \qquad h \circ (g \circ f) \ = \ (h \circ g) \circ f$$

reflects the computational property that for all effectful expressions $\vdash f : A$ and $a : A \vdash g : B$ and $b : B \vdash h : C$, the two effectful expressions (i) and (ii) defined below

$$\boxed{\begin{array}{ll} (i) & \mathbf{let}\ a \stackrel{\varepsilon}{=} f\ \mathbf{in}\, (\mathbf{let}\ b \stackrel{\varepsilon'}{=} g\ \mathbf{in}\ h) \\ (ii) & \mathbf{let}\ b \stackrel{\varepsilon'}{=} (\mathbf{let}\ a \stackrel{\varepsilon}{=} f\ \mathbf{in}\ g)\ \mathbf{in}\ h \end{array}}$$

are equal whenever the *polarities* $\varepsilon, \varepsilon' \in \{\oplus, \ominus\}$ of the **let** constructs are the same. Here, we use the polarity $\varepsilon \in \{\oplus, \ominus\}$ to indicate in which style $\mathbf{let}\ a \stackrel{\varepsilon}{=} u\ \mathbf{in}\ t$ should be evaluated: CBV ($\varepsilon = \oplus$) or CBN ($\varepsilon = \ominus$). The fact that the expressions (i) and (ii) behave in the same way implies in particular that they evaluate f, g and h in the same order in CBV as well as in CBN, as shown below.

Composition style	Order of evaluation	
$(\varepsilon, \varepsilon') = (\oplus, \oplus)$	$(i) = (ii)$	f then g then h
$(\varepsilon, \varepsilon') = (\ominus, \ominus)$	$(i) = (ii)$	h then g then f

Mixing call-by-name and call-by-value. In many concrete situations, the programmer would like to control and reason about the order of evaluation. This can be modelled by letting both styles of **let** constructs appear inside expressions. Inspecting the two effectful expressions (i) and (ii) again in that hybrid scenario, we see that the two expressions (i) and (ii) behave in the same way when $(\varepsilon, \varepsilon') = (\ominus, \oplus)$ but behave differently when $(\varepsilon, \varepsilon') = (\oplus, \ominus)$. In particular, in that latter case, the expression f is evaluated before h and then g in (i) whereas the expression h is evaluated before f and then g in (ii).

Composition style	Order of evaluation	
$(\varepsilon, \varepsilon') = (\ominus, \oplus)$	$(i) = (ii)$	g then f then h
$(\varepsilon, \varepsilon') = (\oplus, \ominus)$	(i)	f then h then g
	(ii)	h then f then g

A natural question is how we could develop a mathematical framework that considers seriously the combination of evaluation paradigms, without a priori biases towards monads nor comonads. In order to reflect these equations, such a framework needs to integrate both Kleisli and co-Kleisli categories, where the former associativity equation holds

$$(h \bullet g) \circ f \quad = \quad h \bullet (g \circ f)$$

but where the latter associativity equation

$$(h \circ g) \bullet f \quad = \quad h \circ (g \bullet f)$$

does not necessarily hold in general. There is no hope of defining categories and we thus need to move to "non-associative" forms of categories. This is the direction taken by the third author [2014b] based on a non-associative and polarized notion of *duploid*.

The idea of non-associativity is far from new: it appeared for the first time in Girard's *"constructive"* classical logic **LC**, which introduced a formal distinction between "positive" and "negative" formulae [Girard 1991]. The idea then resurfaced with the "Blass problem" in game semantics [Blass 1992; Abramsky 2003], whose origin was traced back to the existence of an adjunction between categories of "positive" and "negative" games [Melliès 2005]. However, non-associativity was mainly perceived as an anomaly until the introduction of duploids and their *computational account of adjunctions* where it was shown that having "three fourths" of the associativity equations captures directly effectful computation integrating both monadic and comonadic effects.

1.2 The non-associative category associated to an adjunction

Adjunctions. In order to intertwine the interpretations of CBV and CBN evaluation in a single mathematical structure including the Kleisli and co-Kleisli categories, a good starting point is indeed to consider a pair of adjoint functors

$$\mathscr{A} \underset{R}{\overset{L}{\rightleftarrows}} \mathscr{B} \quad (L \dashv R) \tag{2}$$

Incidentally, shifting attention from Moggi's monads to adjunctions became standard after the pioneering works of Power and Robinson [1997], Fiore [1994], Thielecke [1997], and Benton [1996] in the 1990's, and after Levy's Call by Push Value [1999, 2004, 2005] which built upon these works.

The adjunction induces a monad $T = R \circ L$ on the category $\mathscr{A}$ and a comonad $K = L \circ R$ on the category $\mathscr{B}$. In order to mix the CBV style and the CBN style we need to combine the Kleisli category $\mathbf{Kl}[\mathscr{A}, T]$ and the co-Kleisli category $\mathbf{coKl}[\mathscr{B}, K]$ in a single algebraic structure.

The collage category of an adjunction. It is well-known that an adjunction $L \dashv R$ can equivalently be seen as a bifibration $p : \mathscr{E} \to \mathbf{2}$ over the order category $\mathbf{2} = 0 \to 1$ with two objects 0 and 1 and a unique map $\mathbf{trans} : 0 \to 1$. Here, the category $\mathscr{E} = \mathbf{coll}_{L,R}$ is defined as the *collage* of the adjunction $L \dashv R$: its objects are the pairs $(0, A)$ where A is an object of $\mathscr{A}$ and the pairs $(1, B)$ where B is an object of $\mathscr{B}$, and

- its maps $(0, A) \to (0, A')$ are the maps $A \to A'$ in $\mathscr{A}$,
- its maps $(1, B) \to (1, B')$ are the maps $B \to B'$ in $\mathscr{B}$,
- its maps $(0, A) \to (1, B)$ are the maps $A \to RB$ in $\mathscr{A}$ or equivalently $LA \to B$ in $\mathscr{B}$,
- there are no maps of the form $(1, B) \to (0, A)$.

The bifibration $p : \mathscr{E} \to \mathbf{2}$ transports every object of the form $(0, A)$ to 0 and of the form $(1, B)$ to 1. Note that the category $\mathscr{E}$ comes equipped with two injective on objects and fully faithful functors

$$\mathscr{A} \xrightarrow{inj_{\mathscr{A}}} \mathscr{E} \xleftarrow{inj_{\mathscr{B}}} \mathscr{B}$$

identifying $\mathscr{A}$ and $\mathscr{B}$ as the fibers over 0 and 1 respectively. We find convenient to write A for $(0, A)$ and B for $(1, B)$ when there are no ambiguities. We also call *transverse* a map of the form $f : A \to B$ with image $p(f) = \mathbf{trans}$. Note that every object $A \in \mathscr{A}$ induces a transverse map

$$A_{01} \quad : \quad A \xrightarrow{\mathscr{E}} LA \tag{3}$$

obtained by "pushing" the object $(0, A) \in \mathscr{E}$ along the map $\mathbf{trans} : 0 \to 1$. Symmetrically, every object $B \in \mathscr{B}$ induces a transverse map

$$B_{01} \quad : \quad RB \xrightarrow{\mathscr{E}} B \tag{4}$$

obtained by "pulling" the object $(1, B) \in \mathscr{E}$ along the map $\mathbf{trans} : 0 \to 1$. Note also that every transverse map $f : (0, A) \to (1, B)$ in the collage category can be equivalently be seen as a map $f^{\lhd} : A \to RB$ in $\mathscr{A}$ or as a map $f^{\rhd} : LA \to B$ in $\mathscr{B}$. The two maps $f^{\lhd}$ and $f^{\rhd}$ can be characterized as the unique maps of $\mathscr{E}$ making the diagram below

$$\begin{array}{ccc} RB & \xrightarrow{B_{01}} & B \\ {\scriptstyle f^{\lhd}}\uparrow & \overset{f}{\nearrow} & \uparrow{\scriptstyle f^{\rhd}} \\ A & \xrightarrow[A_{01}]{} & LA \end{array}$$

commutative in the collage category.

The duploid associated to an adjunction. Every adjunction $L \dashv R$ induces a non-associative category $\mathbf{dupl}_{L,R}$ which contains the Kleisli category $\mathbf{Kl}[\mathscr{A}, RL]$ of the monad RL and the Kleisli category $\mathbf{coKl}[\mathscr{B}, LR]$ of the comonad LR as full subcategories. The non-associative category $\mathbf{dupl}_{L,R}$ is constructed in Munch-Maccagnoni [2014b] where it is called a *duploid* because it comes equipped with a polarity structure recalled in §3. The objects of $\mathbf{dupl}_{L,R}$ are the objects $(0, A)$ and $(1, B)$ of the collage category, for $A \in \mathscr{A}$ and $B \in \mathscr{B}$. Its maps are defined as follows. Every object $X = (0, A)$ or $X = (1, B)$ of the duploid induces a transverse map

$$X_{01} \quad : \quad X_0 \xrightarrow{\mathscr{E}} X_1 \tag{5}$$

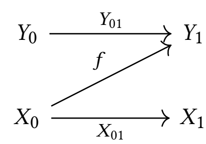
defined as $X_{01} = A_{01}$ in (3) when $X = (0, A)$ and as $X_{01} = B_{01}$ in (3) when $X = (1, B)$. A morphism $f : X \to Y$ between two objects X and Y of the duploid $\mathbf{dupl}_{L,R}$ with associated transverse maps

$$X_{01} \;:\; X_0 \xrightarrow{\mathscr{E}} X_1 \qquad\qquad Y_{01} \;:\; Y_0 \xrightarrow{\mathscr{E}} Y_1$$

is simply defined as a transverse map

$$f \;:\; X_0 \xrightarrow{\mathscr{E}} Y_1$$

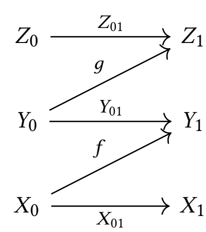
in the collage category $\mathscr{E} = \mathbf{coll}_{L,R}$. The situation may be depicted as follows:



The exercise of defining the composite

$$g \odot f \;:\; X \longrightarrow Z$$

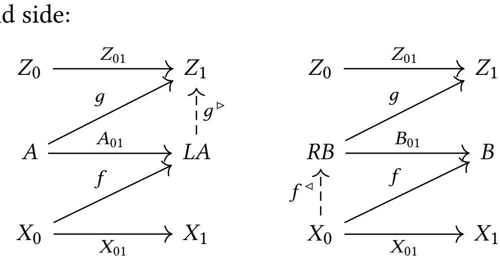
of two morphisms $f : X \to Y$ and $g : Y \to Z$ of the duploid $\mathbf{dupl}_{L,R}$ looks somewhat challenging when one considers the diagram of the adjunction:



in the collage category $\mathscr{E} = \mathbf{coll}_{L,R}$. The composite $g \odot f : X \to Z$ is defined in two different ways, depending on the polarity of the intermediate object Y:

- when $Y = (0, A)$ is positive, the transverse map $Y_{01} = A_{01}$ is of the form (3) and there exists for that reason a unique map $g^{\triangleright} : LA \to Z_1$ such that $g = g^{\triangleright} \circ_{\mathscr{E}} A_{01}$; the composite $g \odot f$ is defined in that case as $g \odot f = g^{\triangleright} \circ_{\mathscr{E}} f : X_0 \to Z_1$ in the collage category.
- when $Y = (1, B)$ is negative, the transverse map $Y_{01} = B_{01}$ is of the form (4) and there exists for that reason a unique map $f^{\triangleleft} : X_0 \to RB$ such that $f = B_{01} \circ_{\mathscr{E}} f^{\triangleright}$; the composite $g \odot f$ is defined in that case as $g \odot f = g \circ_{\mathscr{E}} f^{\triangleleft} : X_0 \to Z_1$ in the collage category.

The recipe used to define the composite $g \odot f : X \to Z$ in the collage category is depicted below, with the case when $Y = (0, A)$ is positive on the lefthand side, and the case when $Y = (0, B)$ is negative on the righthand side:



The construction establishes $\mathbf{dupl}_{L,R}$ as a simple and canonical way to integrate the Kleisli and co-Kleisli categories $\mathbf{Kl}[\mathscr{A}, RL]$ and $\mathbf{coKl}[\mathscr{B}, LR]$ in a larger overarching mathematical structure. Indeed, an easy computation shows that

- $\mathbf{Kl}[\mathscr{A}, RL]$ coincides with the full subcategory of positive objects (= objects of $\mathscr{A}$)
- $\mathbf{coKl}[\mathscr{B}, LR]$ coincides with the full subcategory of negative objects (= objects of $\mathscr{B}$)

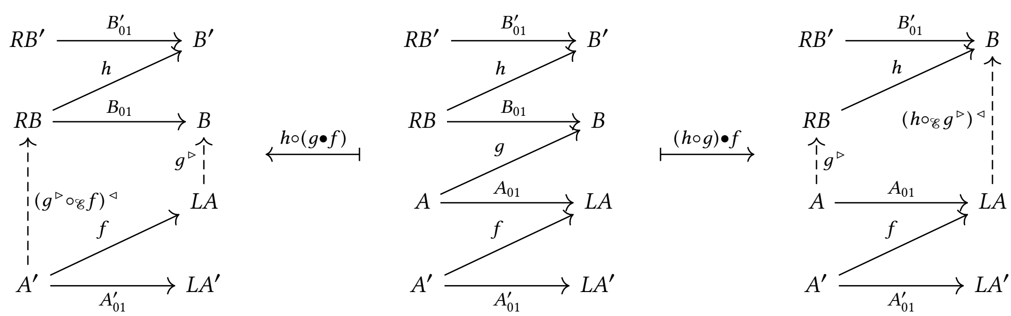
in the non-associative category $\mathbf{dupl}_{L,R}$. For that reason, it makes sense to write the composite $g \odot f$ as $g \bullet f$ when $Y = (0, A)$ is positive, and as $g \circ f$ when $Y = (1, B)$ is negative. The fact that composition is not associative in $\mathbf{dupl}_{L,R}$ comes from the fact that three maps

$$A' \xrightarrow{f} A \xrightarrow{g} B \xrightarrow{h} B'$$

defining a path of length 3 in $\mathbf{dupl}_{L,R}$ may induce different composite maps

$$(i): \quad (h \circ g) \bullet f \qquad \text{and} \qquad (ii): \quad h \circ (g \bullet f)$$

The explicit computation of the two maps in the collage category $\mathscr{C} = \mathbf{coll}_{L,R}$ is depicted below:



The two maps give rise to maps of the collage category

$$(h \circ g) \bullet f = (h \circ_{\mathscr{C}} g^{\triangleright})^{\triangleleft} \circ_{\mathscr{C}} f \qquad h \circ (g \bullet f) = h \circ_{\mathscr{C}} (g^{\triangleright} \circ_{\mathscr{C}} f)^{\triangleleft}$$

which are in general different for good reasons, as we illustrate below.

1.3 Non-associativity seen as a blessing: thunkable and linear maps

The fact that duploids are non-associative forms of categories should not be seen as a defect but rather as an opportunity to characterize two specific classes of maps: thunkable and linear maps, conveying important computational intuitions. It is natural to ask when an expression in an effectful language is pure. One possible definition is that it can be substituted like a value, a notion called *algebraic value* or *thunkable* [Thielecke 1997] expression. One benefit of working in a duploid or more generally in a non-associative category, is that thunkability for a map $f : X \to Y$ can be formulated as an associativity property of composition [Munch-Maccagnoni 2014b] capturing the following syntactic equation:

$$\textbf{let } y \overset{\oplus}{=} f \textbf{ in let } z \overset{\ominus}{=} g_y \textbf{ in } h_z \quad = \quad \textbf{let } z \overset{\ominus}{=} (\textbf{let } y \overset{\oplus}{=} f \textbf{ in } g_y) \textbf{ in } h_z \tag{6}$$

for all expressions g_y, h_z. This leads us to the following definition of thunkable maps as well as the symmetric notion of linear maps in any non-associative category.

Thunkable and linear maps. A map f is called *thunkable* when every path of length 3 that starts with f:

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} W$$

associates, that is, the following holds:

$$h \odot (g \odot f) = (h \odot g) \odot f$$

Dually, a morphism h is called *linear* if every path of length 3 that ends with h associates.

Thunkable and linear maps: illustration. We illustrate the benefits of introducing these concepts with probabilistic computations and the finite distribution monad. Recall that the finite distribution monad $T : Set \to Set$ associates to every set A the set TA of finite probability distributions $\sum_i p_i \mid a_i\rangle$ on A, that is finite families $(a_i)_{i\in I}$ of elements $a_i \in A$ equipped with a function $p : I \to [0,1]$ to the interval $[0,1]$ of reals between 0 and 1, such that $\sum_i p_i = 1$. The Kleisli category Set_T associated to the monad T is the category of *stochastic maps* whose morphisms $f : B \to B'$ between sets B, B' are defined as functions $f : B \to TB'$ which transport every element $b \in B$ to a finite probability distribution $f(b) = \sum_i p_i(b) \mid f_i(b)\rangle$ of B'.

The duploid construction applied to the adjunction between Set and its Kleisli category Set_T

$$Set \underset{R}{\overset{L}{\rightleftarrows}} Set_T \qquad (L \dashv R)$$

induces a duploid whose objects are sets $(0, A)$ and $(1, B)$ annotated with a polarity 0 for positive and 1 for negative.

- its maps $(0, A) \to (0, A')$ are the stochastic maps $A \to A'$,
- its maps $(1, B) \to (1, B')$ are the stochastic maps $TB \to B'$,
- its maps $(0, A) \to (1, B)$ are the stochastic maps $A \to B$,
- its maps $(1, B) \to (0, A)$ are the stochastic maps $TB \to A$.

The composite $g \circ f : X \to Z$ of two morphisms $f : X \to Y$ and $g : Y \to Z$ in the duploid depends on the polarity of the intermediate object Y.

▷ *when* $Y = (0, A)$ *is positive*, we compose the stochastic maps f and g in the usual way, by distributing the probabilities. For instance,

$$\textbf{let } y \overset{\oplus}{=} \sum_i p_i \mid f_i(x)\rangle \textbf{ in } \sum_j q_j \mid g_j(y)\rangle \quad = \quad \sum_{i,j} p_i q_j \mid g_j(f_i(x))\rangle$$

when the coefficients q_j do not depend on y.

▷ *when* $Y = (1, B)$ *is negative*, we substitute the expression $\Sigma_i p_i \mid u_i(x)\rangle$ in the expression $\Sigma_j q_j \mid v_j(y)\rangle$ without distributing the probabilities. For instance,

$$\textbf{let } y \overset{\ominus}{=} \sum_i p_i \mid f_i(x)\rangle \textbf{ in } \sum_j q_j \mid g_j(y)\rangle \quad = \quad \sum_j q_j \Big| g_j\Big(\sum_i p_i \mid f_i(x)\rangle\Big)\Big\rangle$$

when the coefficients q_j do not depend on y.

Now that the duploid associated to the finite distribution monad T has been defined, we characterize the thunkable maps $f : X \to Y$ in the duploid. It is not difficult to see that $f : X \to Y$ is always thunkable when $Y = (1, B)$ is negative. We thus focus on the interesting case where $Y = (0, A)$ is positive. In that case, the map $f : X \to (0, A)$ is a stochastic map to the set A of the general form

$$f(x) := \sum_i p_i(x) \mid f_i(x)\rangle$$

with $\sum_i p_i(x) = 1$ and each $f_i(x) \in A$, for all $x \in X$. At this stage, we introduce the two maps which will play the role of "effectful context" testing the behavior of the map f in the duploid:

$$\begin{array}{c} g : (0, A) \to (1, A) \\ g : a \mapsto 1 \mid a\rangle \end{array} \quad \text{and} \quad \begin{array}{c} h : (1, A) \to (0, TA) \\ h : d \mapsto 1 \mid d\rangle \end{array},$$

defined as (1) the stochastic map $g : A \to A$ which associates to any $a \in A$ the Dirac distribution $1 \mid a\rangle \in TA$ and as (2) the stochastic map $h : TA \to TA$ which associates to any probability distribution $d \in TA$ the Dirac distribution of distributions $1\,|\,d\rangle \in TTA$. This defines a path

$$X \xrightarrow{f} (0, A) \xrightarrow{g} (1, A) \xrightarrow{h} (0, TA)$$

of length 3 which can be composed in two different ways, left-to-right or right-to-left as follows:

$$\begin{array}{lll} \textbf{let } d \overset{\ominus}{=} \big(\textbf{let } a \overset{\oplus}{=} f(x) \textbf{ in } 1\,|\,a\rangle\big) \textbf{ in } 1\,|\,d\rangle & \text{reducing to} & 1\Big|\sum_i p_i(a)\,|\,f_i(a)\rangle\Big\rangle \\ \textbf{let } a \overset{\oplus}{=} f(x) \textbf{ in } \big(\textbf{let } d \overset{\ominus}{=} 1\,|\,a\rangle \textbf{ in } 1\,|\,d\rangle\big) & \text{reducing to} & \sum_i p_i(x)\Big|1\,|\,f_i(x)\rangle\Big\rangle \end{array}$$

When the map f is thunkable, these two expressions must be equal by definition. From this follows that the stochastic map f must transport every element $x \in X$ into a Dirac distribution of the form $1\,|\,b(x)\rangle \in TA$. Conversely, any such map f is thunkable. This shows that in this duploid associated to the probability adjunction, thunkable maps coincide with *values* in the sense of Moggi [1991].

1.4 Continuations, dialogue duploids, and classical notions of computation

One important aspect of duploids is that they exhibit and preserve the perfect symmetry between the monadic and comonadic effects of an adjunction, by treating on an equal footing the CBV and CBN evaluation policies. Our main goal in the present paper is to explore how this symmetric and non-associative account of effects can benefit the long quest for a perfectly symmetric computational account of classical logic, in the spirit and philosophy of the Curry-Howard correspondence.

The self-adjunction of negation. For our purposes, we find convenient and evocative to work with the notion of **dialogue category** introduced by the third author [Melliès 2016; Mellies and Tabareau 2010] as a categorical semantics of linear continuations. Recall that a dialogue category is defined as a symmetric monoidal category $(\mathscr{C}, \otimes, 1)$ equipped with a return object $\perp$ in the following sense:

Definition 1.1. An object $\perp$ is called a *return object* in a symmetric monoidal category $(\mathscr{C}, \otimes, 1)$ when it comes equipped with an object $\perp^A$ and a family of bijections

$$\varphi_{A,B} \quad : \quad \mathscr{C}(A \otimes B, \perp) \quad \xrightarrow{\cong} \quad \mathscr{C}(B, \perp^A)$$

natural in B, for every object A of the category $\mathscr{C}$.

An easy categorical argument shows that every dialogue category comes equipped with a functor

$$\neg \quad : \quad \mathscr{C} \longrightarrow \mathscr{C}^{\mathrm{op}}$$

which transports every object A to the object $\perp^A$ which can be seen as a negation of A and written accordingly as $\neg A$. A well-known fact is that the negation functor defines an adjunction with itself:

$$\mathscr{C} \underset{R = \neg}{\overset{L = \neg}{\rightleftarrows}} \mathscr{C}^{\mathrm{op}} \qquad (L \dashv R) \tag{7}$$

This observation, dating back to Kock [1970], was given emphasis in Thielecke's Ph.D. thesis [1997] on the structure of CPS translations. This self-adjunction plays also a central role in the the foundations of functorial game semantics [Melliès 2012a].

Dialogue duploids. We have seen that the construction of the duploid $\mathbf{dupl}_{L,R}$ associated to an adjunction $L \dashv R$ amounts to building a direct computational interpretation combining the CBV and the CBN models and preserving the symmetry between them. Our purpose in the present paper is to uncover the structural properties of the duploids $\mathbf{dupl}_{L,R}$ associated to a dialogue category. In order to better understand these structures, we start from the symmetric reformulation (up to equivalence) of dialogue categories as *dialogue chiralities* defined below:

Definition 1.2 (Melliès [2012a, 2016]). A **dialogue chirality** is a pair of symmetric monoidal categories $(\mathscr{A}, \oslash, \mathit{true})$ and $(\mathscr{B}, \otimes, \mathit{false})$ equipped with an adjunction $L : \mathscr{A} \rightleftarrows \mathscr{B} : R$ as depicted in (2) together with a symmetric monoidal equivalence:

$$(\mathscr{A}, \oslash, \mathit{true}) \quad \underset{(-)^*}{\overset{(-)^*}{\simeq}} \quad (\mathscr{B}, \otimes, \mathit{false})^{\mathrm{op}} \tag{8}$$

and a family of bijections (called currifications)

$$\chi_{A_1,A_2,B} : \mathscr{A}(A_1 \oslash A_2, RB) \xrightarrow{\cong} \mathscr{A}(A_1, R(A_2^* \otimes B))$$

natural in A_1, A_2 and B and satisfying a coherence diagram.

In order to understand the specific nature of duploids associated to dialogue chiralities, we will develop a general theory of duploids equipped with different forms of monoidal structures, in link with classical logic and linear as well as non linear continuations. In particular, we will define the notion of *dialogue duploid* which describes the structure of a duploid associated to a dialogue chirality.

*A categorical account of Girard's **LC**.* Our work is guided by the observation that dialogue duploids provide a compelling categorical and semantic foundation to the proof-theoretic account of classical logic based on **LC** developed in Girard [1991]. Indeed, it appears that the notion of dialogue duploid provides the categorical semantics of a linear and two-sided multiplicative variant of **LC**, what can be summarized as follows:

Linear two-sided multiplicative **LC**	=	duploids + dialogue chiralities

In that sense, the duploid construction provides in the case of dialogue categories a precise mathematical and denotational counterpart to the multiplicative fragment of the new form of double-negation translation implemented by **LC** which contains the traditional CBV and the CBN computational models as its *positive* and *negative subcategories* respectively.

1.5 The Hasegawa-Thielecke theorem

We have seen in (6) that thunkability of a map $f : X \to Y$ captures a concept of purity which can be expressed by the syntactic equation

$$\mathbf{let}\ y \overset{\oplus}{=} f\ \mathbf{in}\ \mathbf{let}\ z \overset{\ominus}{=} g_y\ \mathbf{in}\ h_z \quad = \quad \mathbf{let}\ z \overset{\ominus}{=} (\mathbf{let}\ y \overset{\oplus}{=} f\ \mathbf{in}\ g_y)\ \mathbf{in}\ h_z$$

for all expressions g_y, h_z. As we will see, this corresponds to an equality between derivations in sequent calculus of the following form:

$$\dfrac{\overset{f}{\Gamma'' \vdash P, \Delta''} \qquad \dfrac{\overset{h_z}{\Gamma, P \vdash N, \Delta} \qquad \overset{g_y}{\Gamma', N \vdash \Delta'}}{\Gamma, \Gamma', P \vdash \Delta, \Delta'}}{\Gamma, \Gamma', \Gamma'' \vdash \Delta, \Delta', \Delta''} \quad = \quad \dfrac{\dfrac{\overset{f}{\Gamma'' \vdash P, \Delta''} \qquad \overset{h_z}{\Gamma, P \vdash N, \Delta}}{\Gamma, \Gamma'' \vdash N, \Delta, \Delta''} \qquad \overset{g_y}{\Gamma', N \vdash \Delta'}}{\Gamma, \Gamma', \Gamma'' \vdash \Delta, \Delta', \Delta''} \tag{9}$$

A weaker concept of purity, *centrality* [Power and Robinson 1997], captures the idea of irrelevance of order of evaluation with another property of commutation (for all $g, h_{x,y}$):

$$\textbf{let } x \overset{\oplus}{=} f \textbf{ in let } y \overset{\oplus}{=} g \textbf{ in } h_{x,y} \quad = \quad \textbf{let } y \overset{\oplus}{=} g \textbf{ in let } x \overset{\oplus}{=} f \textbf{ in } h_{x,y}$$

or in sequent calculus:

$$\dfrac{\overset{f}{\Gamma'' \vdash P, \Delta''} \qquad \dfrac{\overset{g}{\Gamma' \vdash Q, \Delta'} \qquad \overset{h_{x,y}}{\Gamma, P, Q \vdash \Delta}}{\Gamma, \Gamma', P \vdash \Delta, \Delta'}}{\Gamma, \Gamma', \Gamma'' \vdash \Delta, \Delta', \Delta''} \quad = \quad \dfrac{\overset{g}{\Gamma' \vdash Q, \Delta'} \qquad \dfrac{\overset{f}{\Gamma'' \vdash P, \Delta''} \qquad \overset{h_{x,y}}{\Gamma, P, Q \vdash \Delta}}{\Gamma, \Gamma'', Q \vdash \Delta, \Delta''}}{\Gamma, \Gamma', \Gamma'' \vdash \Delta, \Delta', \Delta''} \tag{10}$$

Note that, strikingly, these two instances of commutations between f and g are the same up to duality in the sequent calculus. For the classical notions of computation we are considering, another ingredient makes them actually coincide: the presence of a negation connective which is involutive at the level of proof denotation, whose rules in sequent calculus provide a way to exchange between the left-hand and right-hand sides without loss of information. We formalize this idea with our proof of the Hasegawa-Thielecke theorem (thm. 11.2) both semantically and syntactically, using the linear classical L-calculus (§9).

The conceptually simple proof of the Hasegawa-Thielecke theorem illustrates the benefits of using duploids to reason about effectful programs. This result was noticed as a characterisation of centrality by Thielecke [1997] in the context of categorical semantics for continuations, in which it plays an important role (see Thielecke [1997], Selinger [2001], and Hasegawa and Kakutani [2002] among others). The essential status of thunkability as a concept distinct from centrality became apparent in the works on the direct axiomatic theory of monadic effects by Führmann [2000, 1999]. Our conceptual reformulation of the proof suggests that expressing thunkability as an associativity property is a key part of the result, yet one which remains true and useful beyond continuation models.

It follows from this theorem that in any dialogue category, *a map is thunkable if and only if it is central*. In particular, *the double-negation monad is commutative if and only if it is idempotent*. The latter condition corresponds to the case where the duploid is a category (hence, in the case of cartesian dialogue categories, to a boolean algebra by Joyal's obstruction theorem (§13)). The refinement of this property from the cartesian to the symmetric monoidal setting was suggested by Hasegawa and played a key role in the second author's analysis of the Blass problem in game semantics as a non-commutativity of the double-negation monad [Melliès 2005]. We are not aware of a previously-published proof of this theorem in the symmetric monoidal case.

1.6 Summary and main contributions

After this long but necessary introduction, we explain in §2 how to reason diagrammatically in a non-associative category, and then recall in §3 the notion of duploid. We then start our journey towards the linear classical L-calculus by introducing in §4 and §5 the notion of symmetric monoidal duploid, alongside the crucial notion of *adjunction between graph morphisms* of non-associative categories. We use this notion in §6 to define symmetric monoidal closed duploid and show a correspondence with linear call-by-push-value adjunction models. We also recall in §7 the *linear call-by-push-value L-calculus* of Curien, Fiore, and Munch-Maccagnoni [CFMM 2016] and establish a soundness theorem of the interpretation in symmetric monoidal closed duploid.

From this intuitionistic basis, we consider models with an involutive negation, leading to the notion of dialogue duploid in §8 and to the *linear classical L-calculus* in §9. We show that dialogue duploids are in correspondence with dialogue chiralities and soundly interpret the linear classical L-calculus. Building on this result, we illustrate the relevance and robustness of our approach by defining the syntactic dialogue duploid in §10 and by proving in §11 the Hasegawa-Thielecke theorem using both semantic and syntactic methods. In this extended version of the paper, we also explain in §12 the relationship between the two-sided and the *one-sided* variant of the linear classical L-calculus as a coherence result. We then give a historical perspective in §13 on the different Curry-Howard approaches to classical logic, and finally conclude and give directions for future work in §14.

This extended version contains an appendix with additional illustrations, and some details of proofs. A table of contents and a list of figures are provided on the final page.

2 Non-associative categories

We have observed in the introduction that the composition of effectful programs $g, f \mapsto g \odot f$ is not associative in general when one wants to make positive and negative types coexist in the same overarching mathematical structure. This justifies to develop a good theory and practice of non-associative categories. As we will see, working with non-associative categories is not only possible, it is also a highly compelling exercise, which sheds light on the fundamental nature of effects, commutative or not commutative.

Definition 2.1. A **unital magmoid** or **non-associative category** $\mathcal{M}$ is defined as a reflexive graph equipped with a composition law

$$\odot_{X,Y,Z} \quad : \quad \mathcal{M}(Y,Z) \times \mathcal{M}(X,Y) \longrightarrow \mathcal{M}(X,Y)$$

which associates to every pair of maps $f : X \to Y$ and $g : Y \to Z$ a composite map $g \odot f : X \to Z$, such that the neutrality equations below are satisfied

$$f \odot \mathsf{id}_X = f = \mathsf{id}_Y \odot f$$

for every map $f : X \to Y$, where $\mathsf{id}_X \in \mathcal{M}(X,X)$ denotes the chosen map at object X of the reflexive graph. We use the notation $|\mathcal{M}|$ for the set (or more generally the class) of objects of $\mathcal{M}$.

Given a non-associative category $\mathcal{M}$, we define $\mathcal{M}^{\mathrm{op}}$ as the non-associative category with the same objects as $\mathcal{M}$ and with the orientation of maps reversed, in the sense that $\mathcal{M}^{\mathrm{op}}(X,Y) := \mathcal{M}(Y,X)$.

In a non-associative category $\mathcal{M}$, we declare that a path (f, g, h) of length 3

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} W$$

associates when the associativity equation below is satisfied:

$$(h \odot g) \odot f = h \odot (g \odot f)$$

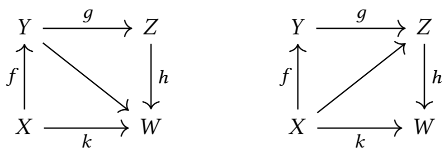
We recall the definitions of linear and thunkable maps given in the introduction:

- A map h is called **linear** when every path of the form (f, g, h) associates,
- A map f is called **thunkable** when every path of the form (f, g, h) associates.

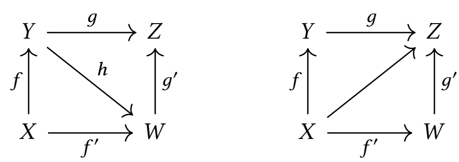
Note also that the usual definitions of epis and monos as right and left cancellable maps in an associative category immediately extend to non-associative categories:

- a map $e : X \to Y$ is called **epi** when $f \odot e = g \odot e$ implies $f = g$ for all maps $f, g : Y \to Z$,
- a map $m : Y \to Z$ is called **mono** when $m \odot f = m \odot g$ implies $f = g$ for all maps $f, g : X \to Y$.

In the same way as reasoning within associative categories can be performed by chasing commutative diagrams, reasoning within non-associative categories generally reduces to chasing "triangulated" commutative diagrams, and rewriting them using the usual and well-studied notion of flip of triangulation [Pournin 2014]. Typically, the fact that the path (f, g, h) associates means that one can "flip" the triangular commutative decomposition of the square on the left into the triangular commutative decomposition of the same square on the right:

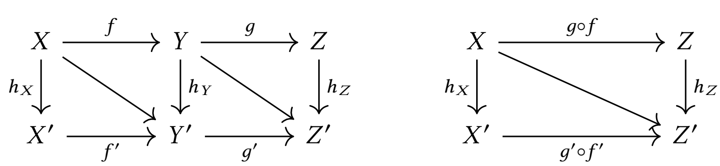


and conversely, from the commutative diagram on the left to the commutative diagram on the right. Similarly, one can flip the commutative diagram on the left into the commutative diagram on the right whenever the path (f, h, g') associates:



The notion of flip between triangular decompositions plays a central role in the combinatorial study of associativity. It is thus striking to see the same combinatorial idea turned here into an equational reasoning method for effectful programs.

At the same time, it is remarkable that very basic principles of usual associative categories are not necessarily true anymore in non-associative categories. Typically, the fact that the triangulated diagram below on the left commutes, in the sense that $h_Y \circledcirc f = f' \circledcirc h_X$ and $h_Z \circledcirc g = g' \circledcirc h_Y$,



(11)

does not imply that the triangulated diagram on the right commutes, in the sense that $h_Z \circledcirc (g \circledcirc f) = (g' \circledcirc f') \circledcirc h_X$. However, the triangulated diagram commutes when the three paths below associate:

$$(h_X, f', g') \qquad\qquad (f, h_Y, g') \qquad\qquad (f, g, h_Z)$$

The property can be established either equationally or diagrammatically, by chasing and flipping commutative triangulations. More details can be found in Appendix B.

3 Duploids

We have seen in the introduction that every adjunction $L \dashv R$ induces a non-associative category $\mathbf{dupl}_{L,R}$ where every object is polarized either positive or negative. In this section, we recall the notion of *duploid* introduced in Munch-Maccagnoni [2014b] as non-associative categories equipped with a polarity structure defined as a pair of shift operators $X \mapsto \Downarrow X$ and $X \mapsto \Uparrow X$. The notion of duploid is justified by the fact, established in Munch-Maccagnoni [2014b], that it characterizes the class of non-associative categories $\mathbf{dupl}_{L,R}$ associated to an adjunction $L \dashv R$, up to an equivalence, as per Thm 3.7 below.

We first recall the observation made in Clairambault and Munch-Maccagnoni [2017] that every non-associative category $\mathcal{M}$ comes with an intrinsic notion of polarity on objects.

Definition 3.1 (Polarity). An object $X \in |\mathcal{M}|$ is called **$\mathcal{M}$-positive** when, for all $Y \in |\mathcal{M}|$, all maps of $\mathcal{M}(X, Y)$ are linear. Symmetrically, an object Y of $\mathcal{M}$ is **$\mathcal{M}$-negative** when, for all $X \in |\mathcal{M}|$, all maps of $\mathcal{M}(X, Y)$ are thunkable.

Note that an object X may be both $\mathcal{M}$-positive and $\mathcal{M}$-negative: this is the case in particular for every object X of an associative category. Note also that, if a map f is linear in the non-associative category $\mathcal{M}$, then f is thunkable in the opposite non-associative category $\mathcal{M}^{\mathrm{op}}$, and conversely. From this follows that $(-)^{\mathrm{op}}$ reverses the polarities.

Definition 3.2. A **positive shift** on a non-associative category $\mathcal{M}$ consists of an object $\Downarrow X$ equipped with a thunkable epi $\omega_X : X \to \Downarrow X$ for every object X, satisfying the following lifting property: for every map $f : X \to Y$, there exists a unique linear map $f^\dagger : \Downarrow X \to Y$ making the diagram (12) on the left commute, that is, $f = f^\dagger \circledcirc \omega_X$. A **negative shift** $(\Uparrow, \delta)$ is a positive shift on $\mathcal{M}^{\mathrm{op}}$.

(12)

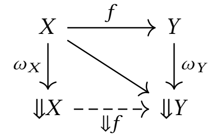
A nice and instructive exercise in non-associative categories is to show that the object $\Downarrow X$ is $\mathcal{M}$-positive, that the lifting $(-)^\dagger$ transports thunkable maps into thunkable maps, and that the map $\overline{\omega}_X := \mathrm{id}_X^\dagger : \Downarrow X \to X$ defined as the lift of the identity map id_X is both linear and thunkable, and satisfies the two equations:

$$\overline{\omega}_X \circledcirc \omega_X = \mathrm{id}_X \qquad \text{and} \qquad \omega_X \circledcirc \overline{\omega}_X = \mathrm{id}_{\Downarrow X}$$

defining a left and right inverse to the map ω_X.

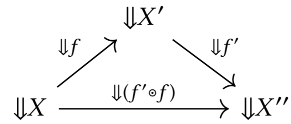
Despite respecting all the usual conditions to be an isomorphism in a usual associative category, the map ω_X should not be considered as an isomorphism in non-associative categories. Indeed, the correct notion for an isomorphism in a non-associative category is an inversible morphism such that the morphism and its inverse are thunkable as well as linear—this reflects and formalises an important observation made on the notion of contextual isomorphisms in Levy [2017]. The objects X and $\Downarrow X$ are isomorphic in this sense exactly when X is positive.

A positive shift defines a function $X \mapsto \Downarrow X$ on objects which can be extended to a function which transports every map $f \in \mathcal{M}(X, Y)$ to the unique map $\Downarrow f \in \mathcal{M}(\Downarrow X, \Downarrow Y)$ making the triangulated diagram below commute:



Note that this unique map $\Downarrow f$ can be defined as the map $\Downarrow f := (\omega_Y \circ f)^\dagger$ obtained by lifting.

The positive shift just defined transports thunkable maps into thunkable maps and preserves identities. On the other hand, it is important to stress that it *does not* preserve composition, in the sense that the functoriality diagram below does not commute in general:



(13)

This lack of functoriality is a direct consequence of the phenomenon observed in (11) that glueing two commutative triangulated squares do not necessarily produce a commutative triangulated square. On the other hand, and this is the whole beauty and simplicity of non-associative categories, one recovers functoriality precisely when a specific path of length 3 associates:

Proposition 3.3. *The diagram (13) commutes precisely when the path below associates*

$$X \xrightarrow{f} X' \xrightarrow{f'} X'' \xrightarrow{\omega_{X''}} \Downarrow X''$$

in the sense that $\omega_{X''} \odot (f' \odot f) = (\omega_{X''} \odot f') \odot f$.

This non-functoriality of the shift operator should not be seen as a defect. On the contrary, it provides a simple combinatorial explanation for an important and subtle phenomenon in effectful programming. This is illustrated in the Appendix C with the example of the duploid associated to the finite distribution monad $T : Set \to Set$, already discussed in the introduction.

At this stage, we are ready to recall (a slight variant of) the definition of duploid from Munch-Maccagnoni [2014b].

Definition 3.4. A **duploid** is a non-associative category equipped with a positive and a negative shift, and where every object is either positive or negative (or both).

Note that $\mathcal{D}^{\mathrm{op}}$ is a duploid whenever $\mathcal{D}$ is a duploid.
Given a duploid $\mathcal{D}$, we find convenient to introduce below notations for usual (associative) subcategories of $\mathcal{D}$:

- $\mathcal{D}_l$ is the subcategory of linear maps,
- $\mathcal{D}_t$ is the subcategory of thunkable maps,
- $\mathscr{P}$ is the full subcategory of $\mathcal{D}$-positive objects,
- $\mathscr{N}$ is the full subcategory of $\mathcal{D}$-negative objects,
- $\mathscr{P}_t$ is the subcategory of thunkable maps of $\mathcal{P}$,
- $\mathscr{N}_l$ is the subcategory of linear maps of $\mathcal{N}$.

Given $f \in \mathcal{M}(X, Y)$ and $g \in \mathcal{M}(Y, Z)$, we find convenient to write $g \odot f$ as $g \bullet f$ when Y is $\mathcal{M}$-positive and as $g \circ f$ when Y is $\mathcal{M}$-negative.

Definition 3.5. A duploid functor $F : \mathcal{D} \to \mathcal{E}$ between duploids consists of a function $F : |\mathcal{D}| \longrightarrow |\mathcal{E}|$ which preserves polarities of objects, together with a family of functions

$$F_{X,Y} \quad : \quad \mathcal{D}(X, Y) \quad \longrightarrow \quad \mathcal{E}(FX, FY)$$

which preserves compositions and identities as well as linearity and thunkability.

Proposition 3.6. *Duploids, duploid functors and thunkable linear natural transformations form a 2-category $\mathcal{D}upl$.*

The notion of duploid is justified in Munch-Maccagnoni [2014b] by the following characterization result:

Theorem 3.7 (Munch-Maccagnoni [2014b]). *Every non-associative category* $\mathbf{dupl}_{L,R}$ *associated to an adjunction $L \dashv R$ comes equipped with a duploid structure, where $\mathscr{P}$ is equivalent to the Kleisli category on the monad $T = R \circ L$, and $\mathscr{N}$ is equivalent to the co-Kleisli category on the comonad $K = L \circ R$. Moreover,* $\mathbf{dupl}_{L,R}$ *is associative if and only if the monad, or equivalently the comonad, is idempotent. Conversely, every duploid $\mathcal{D}$ induces an adjunction*

$$\mathscr{P}_t \underset{R}{\overset{L}{\rightleftarrows}} \mathscr{N}_l \quad (\bot) \tag{14}$$

where $L = \Uparrow$ and $R = \Downarrow$ are defined by the negative and positive shift operators, whose associated duploid $\mathbf{dupl}_{L,R}$ *is equivalent to $\mathcal{D}$.*

4 Symmetric monoidal duploids

We have just seen how the notion of duploid enables one to characterize the non-associative categories associated to an adjunction $L \dashv R$ (thm. 3.7). We want to extend this characterisation to relate adjunctions giving rise to continuation models to a duploidal axiomatisation of classical (linear) logic reflecting the dualities of sequent calculus. In this section we start with the general case of the structure inherited by a duploid $\mathbf{dupl}_{L,R}$ associated to an adjunction $L \dashv R$ of the form (2) where the category $\mathscr{A}$ is equipped with a symmetric monoidal structure $(\mathscr{A}, \oslash, \mathit{true})$ and where the monad $T = R \circ L$ is strong.

A monad on such a category is **strong** if it is equipped with a pair of left and right strengths related by symmetry:

$$\mathrm{rstr}_{A_1,A_2} : TA_1 \oslash A_2 \longrightarrow T(A_1 \oslash A_2) \qquad \mathrm{lstr}_{A_1,A_2} : A_1 \oslash TA_2 \longrightarrow T(A_1 \oslash A_2)$$

In that case, the Kleisli category $\mathbf{Kl}[\mathscr{A}, T]$ comes equipped with a premonoidal structure compatible with the original tensor product. The tensor product $f \ltimes A_2$ of a Kleisli map $f : A_1 \to TA_1'$ and an object A_2 is defined as

$$f \ltimes A_2 \quad : \quad A_1 \oslash A_2 \xrightarrow{f \oslash A_2} TA_1' \oslash A_2 \xrightarrow{\mathrm{rstr}} T(A_1' \oslash A_2)$$

and symmetrically, the tensor product of an object A_1 and a Kleisli map $g : A_2 \to TA_2'$ is defined as

$$A_1 \rtimes g \quad : \quad A_1 \oslash A_2 \xrightarrow{A_1 \oslash g} A \oslash TA_2' \xrightarrow{\mathrm{lstr}} T(A_1 \oslash A_2')$$

The compatibility between the monoidal structure on $\mathscr{A}$ and the premonoidal structure on $\mathbf{Kl}[\mathscr{A}, T]$ is witnessed by the fact that the identity-on-object functor $\iota : \mathscr{A} \to \mathbf{Kl}[\mathscr{A}, T]$ transports (strictly) the symmetric monoidal structure of $\mathscr{A}$ to the symmetric premonoidal structure of $\mathbf{Kl}[\mathscr{A}, T]$. Recall that given two maps $f : A_1 \to A_1'$ and $g : A_2 \to A_2'$ in a premonoidal category $\mathscr{P}$, the diagram

$$\begin{array}{ccc} A_1 \otimes A_2 & \xrightarrow{f \ltimes A_2} & A_1' \otimes A_2 \\ {\scriptstyle A_1 \rtimes g}\downarrow & & \downarrow{\scriptstyle A_1' \rtimes g} \\ A_1 \otimes A_2' & \xrightarrow{f \ltimes A_2'} & A_1' \otimes A_2' \end{array}$$

does not necessarily commute. A map f is called *central* when this diagram commutes for all maps g. One shows that the functor ι transports every morphism in $\mathscr{A}$ into a central morphism in $\mathbf{Kl}[\mathscr{A}, T]$. The importance of this structure in the semantics on effects has been recognised and intensively studied [Power and Robinson 1997; Power 2002; Staton 2014].

Definition 4.1. A **symmetric monoidal Freyd structure** (also called symmetric premonoidal $[\to, \mathit{Set}]$-category in Power [2002])

$$\iota \quad : \quad (\mathscr{M}, \otimes, 1) \ \to \ (\mathscr{P}, \otimes, 1)$$

is given by a symmetric premonoidal category $(\mathscr{P}, \otimes, 1)$, a symmetric monoidal category $(\mathscr{M}, \otimes, 1)$, and an identity-on-object functor $\iota : \mathscr{M} \to \mathscr{P}$ which transports (strictly) the symmetric monoidal structure of $\mathscr{M}$ to the symmetric premonoidal structure of $\mathscr{P}$, and such that every morphism $\iota(f) : A \to A'$ in $\mathscr{P}$ coming from a morphism $f : A \to A'$ in $\mathscr{M}$ is central in $\mathscr{P}$.

We have seen in the reconstruction with thm. 3.7 of a given duploid $\mathcal{D}$, the positive subcategory $\mathscr{P}$ of $\mathcal{D}$ plays the role of the Kleisli category $\mathbf{Kl}[\mathscr{A}, T]$, whereas the subcategory $\mathscr{P}_t$ of thunkable morphisms of $\mathscr{P}$ plays the role of the category $\mathscr{A}$. This leads us to the following definition:

Definition 4.2. A (positive) **symmetric monoidal structure** $(\otimes, 1)$ on a duploid $\mathcal{D}$ is given by a symmetric monoidal Freyd structure $(\mathscr{P}_t, \otimes, 1) \to (\mathscr{P}, \otimes, 1)$ for the inclusion functor $\mathscr{P}_t \hookrightarrow \mathscr{P}$.

The De Morgan duality of classical logic will also imply to consider later on the dual notion of *negative* symmetric monoidal duploid structure $(\mathcal{D}, ⅋, \bot)$, that is, a symmetric monoidal Freyd category structure $(\mathcal{N}_l, ⅋, \top) \to (\mathcal{N}, ⅋, \top)$ for the inclusion functor $\mathcal{N}_l \hookrightarrow \mathcal{N}$.

The notion of symmetric monoidal duploid is justified by the following theorem:

Theorem 4.3. *Every non-associative category* $\mathbf{dupl}_{L,R}$ *associated to an adjunction* $L : \mathcal{A} \rightleftarrows \mathcal{B} : R$ *where* $\mathcal{A}$ *is symmetric monoidal and the monad* $T = R \circ L$ *is strong comes equipped with a (positive) symmetric monoidal duploid structure. Conversely, every (positive) symmetric monoidal duploid* $\mathcal{D}$ *induces an adjunction* (14) *where* $\mathcal{P}_t$ *is equipped with a symmetric monoidal structure for which the associated monad on* $\mathcal{P}_t$ *is strong.*

5 Graph morphisms and adjunctions between them

In this more technical section, we will describe the structure induced in a symmetric monoidal duploid by its symmetric monoidal Freyd category $(\mathcal{P}_t, \otimes, 1) \to (\mathcal{P}, \otimes, 1)$ on the rest of the duploid. The intuition is that a sequent $A_1, \ldots, A_n \vdash B$ will be interpreted by a duploid morphism $A_1 \otimes \cdots \otimes A_n \to B$.

By using the positive shift $\Downarrow$, we can generalize the premonoidal structure on $\mathcal{P}$ to all objects and morphisms of $\mathcal{D}$. However, this extended tensorial structure is not simply premonoidal, as it inherits the non-functoriality of the shift. We are therefore going to consider (reflexive) *graph morphisms* (notation $F : \mathcal{G} \rightharpoonup \mathcal{H}$), which map objects to objects and morphisms to morphisms while preserving source, target and identities, but not necessarily composition. We now introduce a notion of binoidal graph to describe the tensorial structure on $\mathcal{D}$.

The asynchronous product $\mathcal{G} \boxtimes \mathcal{H}$ of two reflexive graphs $\mathcal{G}$ and $\mathcal{H}$ is defined as the reflexive graph whose objects are pairs (X, Y) of objects X of $\mathcal{G}$ and Y of $\mathcal{H}$ and whose maps are of the form

$$(f, Y) : (X, Y) \to (X', Y) \qquad (X, g) : (X, Y) \to (X, Y')$$

with the maps (id_X, Y) and (X, id_Y) identified and defining the identity map $\mathrm{id}_{(X,Y)}$ of the object (X, Y). A binoidal graph $\mathcal{G}$ is defined as a reflexive graph equipped with a graph morphism

$$\otimes \quad : \quad \mathcal{G} \boxtimes \mathcal{G} \longrightharpoonup \mathcal{G}$$

We write $f \ltimes Y : X \otimes Y \to X' \otimes Y$ and $X \rtimes g : X \otimes Y \to X \otimes Y'$ the image of (f, Y) and (X, g), respectively.

One important observation is that every symmetric monoidal duploid in the sense of def. 4.2 comes equipped with a binoidal structure on objects of any polarity. The tensor product is extended to every pair of objects X and Y as the tensor product of their positive shifts:

$$X \otimes Y \quad := \quad \Downarrow X \otimes^{+} \Downarrow Y \tag{15}$$

Accordingly, writing $\rtimes^{+}$ and $\ltimes^{+}$ for the premonoidal structure between positive objects in $\mathcal{P}$, given a map $f : X \to X'$ and an object Y, we define $f \ltimes Y \ = \ \Downarrow f \ltimes^{+} \Downarrow Y$, and symmetrically for $\rtimes$.

One side-consequence of the definition is that shifting positively coincides in the monoidal duploid $(\mathcal{D}, \otimes, 1)$ with the operation of tensoring with the unit 1, up to a thunkable and linear isomorphism, what can be written:

$$\Downarrow X \quad \cong \quad X \otimes 1.$$

In preparation for the Hasegawa-Thielecke theorem in §11, we establish that every symmetric monoidal duploid $(\mathcal{D}, \otimes, 1)$ still satisfies in this new sense the following property:

Proposition 5.1. *Every thunkable map is central.*

Indeed, the positive shift $\Downarrow$ preserves thunkability, so $\Downarrow f$ is also thunkable and thus, central for $\otimes^{+}$.

The converse property is not true in general: consider for instance the symmetric monoidal duploid $(\mathcal{D}, \otimes, 1)$ associated to the finite distribution monad $T : \mathit{Set} \to \mathit{Set}$ already introduced in §1.3, which maps every set A to the set TA of its finite probability distributions. The monad T is commutative, and every map in $\mathcal{D}$ is thus central. On the other hand, as proved in §1.3, maps into positive objects are thunkable if and only they are of the form $x \mapsto 1 \,|\, b(x)\rangle$.

Working with graph morphisms is particularly fruitful, as we will see using the next notion of adjunction between graph morphisms. As a preliminary observation, note that the composition in any non-associative category $\mathcal{D}$ defines a *graph hom-morphism* $\mathcal{D}^{\mathrm{op}} \boxtimes \mathcal{D} \rightharpoonup \mathit{Set}$.

Definition 5.2. Let $F : \mathcal{D} \rightharpoonup \mathcal{E}$ and $G : \mathcal{E} \rightharpoonup \mathcal{D}$ two graph morphisms between non-associative categories. An **adjunction between graph morphisms** (notation $F : \mathcal{D} \rightleftharpoons \mathcal{E} : G$) is given by an isomorphism of graph morphisms $\mathcal{D}^{\mathrm{op}} \boxtimes \mathcal{E} \rightharpoonup \mathit{Set}$:

$$\varphi \quad : \quad \mathcal{E}(F-,=) \quad \xrightarrow{\cong} \quad \mathcal{D}(-,G=)$$

natural component-wise, that is to say

$$Gg \circledcirc \varphi(f) = \varphi(g \circledcirc f)$$
$$\varphi(f) \circledcirc h = \varphi(f \circledcirc Fh)$$

for every $f \in \mathcal{E}(FX, Y)$ and every morphisms g of $\mathcal{E}$ and h of $\mathcal{D}$.

It is an instructive exercise to check the following:

Proposition 5.3. *Let such an adjunction between graph morphisms $\mathcal{E}(F-,=) \cong \mathcal{D}(-,G=)$.*

1. *F preserves thunkability, and G preserves linearity.*
2. *For f, g morphisms in $\mathcal{E}$, one has $G(f \circledcirc g) = Gf \circledcirc Gg$ if and only if $f \circledcirc g \circledcirc \varepsilon_X$ associates where $\varepsilon_X \stackrel{\mathrm{def}}{=} \varphi^{-1}(\mathrm{id}_{GX})$ (for instance, whenever f is linear or the domain of g is negative). (And dually for F and $\eta_X = \varphi(\mathrm{id}_{FX})$.)*

The first property can be thought of as a property of focusing as seen in proof search: left-invertible connectives are focused on the right-hand side of $\vdash$, whereas right-invertible connectives are focused on the left-hand side of $\vdash$. We have previously described an example of the second property and given computational intuitions in the case of $F = \Downarrow$ and $\eta = \omega$. This leads us to our first examples of adjunctions:

Proposition 5.4. *A duploid $\mathcal{D}$ is the same thing as a non-associative category $\mathcal{D}$ where all objects are either positive or negative (or both), together with a left adjoint $\Downarrow$ and a right adjoint $\Uparrow$ to the identity graph morphism $\mathrm{Id}_{\mathcal{D}}$, such that for every object $A \in \mathcal{D}$, $\Downarrow A$ is positive and $\Uparrow A$ is negative.*

6 Symmetric monoidal closed duploids

Important logical connectives (such as the implication) turn out to give rise to adjunctions, and can moreover be characterised in this way, as we show by investigating the closed structure on symmetric monoidal duploids.

Definition 6.1. A (positive) symmetric monoidal duploid is **closed** when each graph morphism $- \otimes A : \mathcal{D} \rightharpoonup \mathcal{D}$ for $A \in \mathcal{D}$ has a right adjoint $A \multimap -$, such that every object $A \multimap B$ is negative.

From this definition we recover a standard model of linear call by value: every symmetric monoidal closed structure on a duploid $\mathcal{D}$ gives rise to a closed structure on the symmetric monoidal Freyd structure of its positive category, in the sense of Power [2002], namely a right adjoint to the functor $\iota- \otimes P : \mathcal{P}_t \to \mathcal{P}$ for every object P of P_t. The closed Freyd structure is obtained with the definition $P \multimapdotinv^+ - \stackrel{\text{def}}{=} \Downarrow(P \multimapdotinv \Uparrow -)$, recalling the one for call-by-value arrows in polarised logics and in call-by-push-value.

However, not every closed duploid structure can be recovered from its symmetric monoidal closed Freyd structure with the converse definition $\Uparrow(P \multimapdotinv^+ \Downarrow N)$. Symmetric monoidal closed duploids correspond in fact to the more general notion of *linear effect calculus* model from Melliès [2012b], or *linear call-by-push-value* (without sums) from Curien, Fiore, and Munch-Maccagnoni [CFMM 2016]—a linear version of the *effect calculus* model of Egger, Møgelberg, and Simpson [2012]. The next definition uses the notions of presheaf-enriched category and adjunction.

Definition 6.2 ([Melliès 2012b; CFMM 2016]). A *linear effect adjunction* is given by a symmetric monoidal category $\mathscr{A}$, an $\widehat{\mathscr{A}}$-category $\underline{\mathscr{B}}$, a $\widehat{\mathscr{A}}$-adjunction $\underline{L} : \underline{\mathscr{A}} \rightleftarrows \underline{\mathscr{B}} : \underline{R}$, and powers of representable presheaves on $\underline{\mathscr{B}}$.

We also recall the following characterisation of linear effect adjunctions given in Melliès [2012b] which less common, but more convenient in this context as we will see.

Proposition 6.3 ([Melliès 2012b]). *A linear effect adjunction is the same thing as a symmetric monoidal category $\mathscr{A}$ together with an adjunction $L : \mathscr{A} \rightleftarrows \mathscr{B} : R$, a pseudo-action $\multimapdotinv : \mathscr{A}^{\mathrm{op}} \times \mathscr{B} \to \mathscr{B}$ of $\mathscr{A}^{\mathrm{op}}$ on $\mathscr{B}$, and a family of adjunctions*

$$L(- \otimes A) : \mathscr{A} \rightleftarrows \mathscr{B} : R(A \multimapdotinv -) \ . \tag{16}$$

Suppose given such a linear effect adjunction, and write $\mathcal{D} = \mathbf{dupl}_{L,R}$ for the associated symmetric monoidal duploid, with $\otimes_{\mathcal{D}}$ its premonoidal tensor product extended to a graph morphism $\mathcal{D} \boxtimes \mathcal{D} \rightharpoonup \mathcal{D}$ as previously. Define for any $A, B \in \mathcal{D}$ the negative object $A \multimapdotinv_{\mathcal{D}} B \stackrel{\text{def}}{=} \Downarrow A \multimapdotinv \Uparrow B$. We observe that the family of bijections between hom-sets underlying (16) is exactly a family of bijections (with $A, B, C \in \mathcal{D}$):

$$\mathcal{D}(A \otimes_{\mathcal{D}} B, C) \cong \mathcal{D}(A, B \multimapdotinv_{\mathcal{D}} C)$$

This family of bijections extends, for each $B \in \mathcal{D}$, into an adjunction $- \otimes_{\mathcal{D}} B : \mathcal{D} \rightleftharpoons \mathcal{D} : B \multimapdotinv_{\mathcal{D}} -$. This follows from a general principle:

Proposition 6.4. *Let $\mathcal{D}, \mathcal{E}$ two non-associative categories and F a graph morphism $\mathcal{E} \rightharpoonup \mathcal{D}$ that* preserves thunkability. *Assume that for each $A \in \mathcal{E}$ there exists a* negative *object $N \in \mathcal{D}$ and a family of bijections $\mathcal{D}(FA, B) \cong \mathcal{E}(A, N)$ natural in A. Then these families of bijections assemble into a right adjoint $F : \mathcal{E} \rightleftharpoons \mathcal{D} : G$.*

Theorem 6.5. *For every linear effect adjunction $L : \mathscr{A} \rightleftarrows \mathscr{B} : R$, its associated non-associative category $\mathbf{dupl}_{L,R}$ comes equipped with a symmetric monoidal closed duploid structure. Conversely, every symmetric monoidal closed duploid $\mathcal{D}$ induces an adjunction $L \dashv R$ equipped with the structure of a linear effect adjunction.*

Example 6.6. *Set* with the finite distribution monad, or any symmetric monoidal closed category $\mathscr{A}$ with a strong monad T, gives rise to linear effect adjunctions $F_T \dashv G_T : \mathscr{A}_T \to \mathscr{A}$ and $F^T \dashv U^T : \mathscr{A}^T \to \mathscr{A}$ [Melliès 2017b; CFMM 2016], and thus to symmetric monoidal closed duploids.

7 The linear call-by-push-value L-calculus

L-calculi are λ-calculi (higher-order rewriting systems) that present themselves in the form of abstract machines (with an explicit context), and whose typing rules closely match those of sequent calculus, subsuming the rich relationship between CPS, abstract machines, focusing, etc. They are the final ingredient in our correspondence.

$$
\begin{array}{llcl}
\text{Values:} & V, W & ::= & a \mid \mu\alpha^{-}.\mathrm{c} \mid () \mid V \otimes W \mid \mu(a\cdot\beta).\mathrm{c} \\
\text{Expressions:} & t, u & ::= & V \mid \mu\alpha^{+}.\mathrm{c} \\
\text{Stacks:} & S & ::= & \alpha \mid \tilde{\mu}a^{+}.\mathrm{c} \mid \tilde{\mu}().\mathrm{c} \mid \tilde{\mu}(a\otimes b).\mathrm{c} \mid V\cdot S \\
\text{Contexts:} & e & ::= & S \mid \tilde{\mu}a^{-}.\mathrm{c} \\
\text{Commands:} & \mathrm{c} & ::= & \langle V \,\|\, e\rangle^{-} \mid \langle t \,\|\, S\rangle^{+}
\end{array}
$$

(a) Grammar

$$
\begin{array}{llll}
(R\tilde{\mu}^{\varepsilon}) & \langle V \,\|\, \tilde{\mu}a^{\varepsilon}.\mathrm{c}\rangle^{\varepsilon} \rhd_R c[V/a] & (E\tilde{\mu}^{\varepsilon}) & e \rhd_E \tilde{\mu}a^{\varepsilon}.\langle a \,\|\, e\rangle^{\varepsilon} \\
(R\mu^{\varepsilon}) & \langle \mu\alpha^{\varepsilon}.\mathrm{c} \,\|\, S\rangle^{\varepsilon} \rhd_R c[S/\alpha] & (E\mu^{\varepsilon}) & t \rhd_E \mu\alpha^{\varepsilon}.\langle t \,\|\, \alpha\rangle^{\varepsilon} \\
(R1) & \langle () \,\|\, \tilde{\mu}().\mathrm{c}\rangle^{+} \rhd_R c & (E1) & S \rhd_E \tilde{\mu}().\langle () \,\|\, S\rangle^{+} \\
(R\otimes) & \langle V\otimes W \,\|\, \tilde{\mu}(a\otimes b).\mathrm{c}\rangle^{+} \rhd_R c[V/a, W/b] & (E\otimes) & S \rhd_E \tilde{\mu}(a\otimes b).\langle a\otimes b \,\|\, S\rangle^{+} \\
(R\multimap) & \langle \mu(a\cdot\beta).\mathrm{c} \,\|\, V\cdot S\rangle^{-} \rhd_R c[V/a, S/\alpha] & (E\multimap) & V \rhd_E \mu(a\cdot\beta).\langle V \,\|\, a\cdot\beta\rangle^{-}
\end{array}
$$

(b) Reduction and expansion rewriting rules

$$
\mathrm{c} : (\Gamma \vdash \Delta) \qquad \Gamma \vdash t : A \mid \Delta \qquad \Gamma \mid e : A \vdash \Delta
$$

$$
\text{where} \qquad \Gamma = x_1 : A_1, \ldots, x_n : A_n \qquad \Delta = \alpha_1 : B_1, \ldots, \alpha_m : B_m
$$

(c) Judgements

$$
\frac{}{a : A \vdash a : A \mid}\ (\vdash \mathbf{ax}) \qquad \frac{}{\mid \alpha : A \vdash \alpha : A}\ (\mathbf{ax} \vdash)
$$

$$
\frac{\mathrm{c} : (\Gamma, a : A_{\varepsilon} \vdash \Delta)}{\Gamma \mid \tilde{\mu}a^{\varepsilon}.\mathrm{c} : A_{\varepsilon} \vdash \Delta}\ (\tilde{\boldsymbol{\mu}}^{\varepsilon} \vdash) \qquad \frac{\mathrm{c} : (\Gamma \vdash \alpha : A_{\varepsilon}, \Delta)}{\Gamma \vdash \mu\alpha^{\varepsilon}.\mathrm{c} : A_{\varepsilon} \mid \Delta}\ (\vdash \boldsymbol{\mu}^{\varepsilon}) \qquad \frac{\Gamma \mid e : A_{\varepsilon} \vdash \Delta \qquad \Gamma' \vdash t : A_{\varepsilon} \mid \Delta'}{\langle t \,\|\, e\rangle^{\varepsilon} : (\Gamma, \Gamma' \vdash \Delta, \Delta')}\ (\mathbf{cut}^{\varepsilon})
$$

For any $\sigma \in \Sigma(\Gamma, \Gamma')$ and $\forall\tilde{\sigma} \in \Sigma(\Delta, \Delta')$:

$$
\frac{\Gamma \vdash t : A \mid \Delta}{\Gamma' \vdash t[\sigma, \tilde{\sigma}] : A \mid \Delta'}\ (\vdash \sigma, \tilde{\sigma}) \qquad \frac{\Gamma \mid e : A \vdash \Delta}{\Gamma' \mid e[\sigma, \tilde{\sigma}] : A \vdash \Delta'}\ (\sigma, \tilde{\sigma} \vdash) \qquad \frac{\mathrm{c} : (\Gamma \vdash \Delta)}{\mathrm{c}[\sigma, \tilde{\sigma}] : (\Gamma' \vdash \Delta')}\ (\sigma, \tilde{\sigma})
$$

$$
\frac{}{\vdash () : 1 \mid}\ (\vdash 1) \qquad \frac{\mathrm{c} : (\Gamma \vdash \Delta)}{\Gamma \mid \tilde{\mu}().\mathrm{c} : 1 \vdash \Delta}\ (1 \vdash)
$$

$$
\frac{\Gamma \vdash V : A \mid \Delta \qquad \Gamma' \vdash W : B \mid \Delta'}{\Gamma, \Gamma' \vdash V\otimes W : A \otimes B \mid \Delta, \Delta}\ (\vdash_f \otimes) \qquad \frac{\mathrm{c} : (\Gamma, a : A, b : B \vdash \Delta)}{\Gamma \mid \tilde{\mu}(a\otimes b).\mathrm{c} : A \otimes B \vdash \Delta}\ (\otimes \vdash)
$$

$$
\frac{\mathrm{c} : (\Gamma, a : A \vdash \beta : B, \Delta)}{\Gamma \vdash \mu(a\cdot\beta).\mathrm{c} : A \multimap B \mid \Delta}\ (\vdash \multimap) \qquad \frac{\Gamma \vdash V : A \mid \Delta \qquad \Gamma' \mid S : B \vdash \Delta'}{\Gamma, \Gamma' \mid V\cdot S : A \multimap B \vdash \Delta, \Delta'}\ (\multimap \vdash_f)
$$

(d) Typing rules

Fig. 1. Syntax of the linear call-by-push-value L-calculus [CFMM 2016]

In this section, we recall the *linear call-by-push-value L*-calculus introduced in [CFMM 2016] under the name $\mathbf{IMLL}_p^\eta$. It is presented in fig. 1. Later on (§9), we will extend this calculus with a involutive negation to obtain a linear classical sequent calculus.

Syntax. The terms of the L-calculus come in five syntactic categories: expressions, values, contexts, stacks and commands. Values V and stacks S are particular expressions and contexts, respectively, which can be understood as pure or effect-free. Variables (noted a, b, c, ...) stand for values V, and dually, co-variables (noted α, β, γ, ...) stand for stacks S. Types are as follows, each one equipped with a polarity, including atoms X^+, X^-:

$$\text{Types } A, B, A_\varepsilon ::= \underbrace{X^+ \mid 1 \mid A \otimes B}_{\text{positive } (P,Q,A_+)} \mid \underbrace{X^- \mid A \multimap B}_{\text{negative } (N,M,A_-)}$$

Variables are bound by $\tilde{\mu}$ to form a stack $\tilde{\mu}a^+.c$ when the variable a has a positive type, and to form a context $\tilde{\mu}a^-.c$ when the variable a has a negative type. Dually, co-variables are bound by μ to form a value $\mu\alpha^-.c$ when the co-variable α has a negative type, or an expression $\mu\alpha^+.c$ when the co-variable α has a positive type. The value () and the nullary binder $\tilde{\mu}().c$ are associated with the unit of the conjunction 1. We can construct conjunctive terms with either the binary binder $\tilde{\mu}(a\otimes b).c$ or the constructor $V\otimes W$. For the closure, we have the binary binder $\mu(a\cdot\beta).c$ and the constructor $V\cdot S$.

Reductions and expansions. Figure 1 defines a reduction relation $\triangleright_R$ (β-like) and an expansion relation $\triangleright_E$ (η-like) between terms. First, the toplevel reduction $\triangleright_R$ defines an operational semantics because its (well-typed) normal forms for terms without free variables are of the form $\langle V \parallel \alpha\rangle$, denoting a computation ending with a value V.

We note $\to_{RE}$ the contextual closure of ($\beta\eta$) *reduction* $\triangleright_R \cup \triangleleft_E$, and $\simeq_{RE}$ the symmetric, transitive and reflexive closure of $\to_{RE}$. This rewriting theory is well-behaved and meaningful, as follows from standard techniques adapted to the L-calculus [Curien and Munch-Maccagnoni 2010; Munch-Maccagnoni 2017]: for instance, $\to_R$ is confluent and strongly normalising on typed terms, and the $\to_R$-normal, $\to_E$-long forms correspond to focused proofs in the proof-theoretic sense. No additional commuting conversion is necessary.

Example with the composition. In this calculus, the **let** construct is defined as:

$$\mathbf{let}\ a \overset{\varepsilon}{=} t\ \mathbf{in}\ u \quad \overset{\text{def}}{=} \quad \mu\alpha^{\varepsilon'}.\langle t \parallel \tilde{\mu}a^\varepsilon.\langle u \parallel \alpha\rangle^{\varepsilon'}\rangle^\varepsilon .$$

The behavior of this **let** matches the intuitive account given in the introduction. The positive **let** ($\varepsilon = +$) follows the call-by-value paradigm (first reducing t until obtaining a value, then making the substitution in u) whereas the negative **let** ($\varepsilon = -$) follows the call-by-name paradigm (substituting t in u). This follows from the fact that ε determines whether the inner command is of the form $\langle V \parallel e\rangle$ or $\langle t \parallel S\rangle$. Whether the whole expression $\mathbf{let}\ a \overset{\varepsilon}{=} t\ \mathbf{in}\ u$ is evaluated strictly or lazily depends on the polarity ε', which determines whether the whole expression is a value or not.

Typing derivations. The judgements are of the form $\Gamma \vdash \Delta$. As is usual for L-calculi, expressions and values have a type taken from the right-hand side of $\vdash$ while contexts and stacks have a type taken from the left-hand side of $\vdash$. Commands do not have any type. A context on the left-hand side Γ (resp. context on the right-hand side Δ) is a map from an ordered finite set of variables (resp. co-variables) to types of any polarity. The notations Γ, Γ' and Δ, Δ' imply that the contexts have disjoint domains. In the case of the linear call-by-push-value L-calculus, we can in fact see (by induction) that *there is always exactly one formula on the right-hand-side of* $\vdash$. This will no longer be true in the linear classical L-calculus from §9.

Structural rules and linearity. A single structural rule is given for each kind of judgement, as in [CFMM 2016] (see also Atkey [2006]). Structural rules let us *rename* the (co-)variables of the contexts and *reorder* them. To this effect, we define $\mathbf{\Sigma(\Gamma,\Gamma')}$ the set of *bijective* maps $\sigma : \mathrm{dom}\,\Gamma \to \mathrm{dom}\,\Gamma'$ that preserve types (*i.e.* $\Gamma'(\sigma(a)) = \Gamma(a)$ for all $a \in \mathrm{dom}\,\Gamma$). The *(non-linear) call-by-push-value L-calculus* (without sums) is defined by simply relaxing the bijection requirement for structural maps $\sigma \in \Sigma(\Gamma,\Gamma')$—thus allowing *weakening* and *contraction*.

Theorem 7.1 (Subject reduction [CFMM 2016; Munch-Maccagnoni 2017]). *If* $\mathrm{c} \to_{RE} \mathrm{c}'$ *and* $\mathrm{c} : (\Gamma \vdash \Delta)$*, then* $\mathrm{c}' : (\Gamma \vdash \Delta)$*.*

Recovering non-focused rules. Non-restricted constructors $t \otimes u$ and $t \cdot e$, and corresponding non-focused rules $(\vdash \otimes)$ and $(\multimap \vdash)$, can be defined from $V \otimes W$, $V \cdot S$, $(\vdash_f \otimes)$ and $(\multimap \vdash_f)$ by introducing cuts. The constructions are detailed in [CFMM 2016]. These definitions imply the existence of generalized substitutions $\mathrm{c}[t/a]$ and $\mathrm{c}[e/\alpha]$ which appears in the characterisation of thunkability.

Soundness of the calculus.

Theorem 7.2 (Soundness for symmetric monoidal closed duploids). *Given some assigment ρ of atoms to objects respecting the polarity, the interpretation of typed terms of the linear call-by-push-value L-calculus in any symmetric monoidal closed duploid, sending sequents $A_1, \dots, A_n \vdash B$ to hom-sets $\mathcal{D}(A_1^\rho \otimes \cdots \otimes A_n^\rho, B^\rho)$, is invariant modulo typed reductions and expansions.*

The soundness of an interpretation into linear effect adjunctions was previously established in [CFMM 2016]. In fact:

Theorem 7.3. *The interpretation of the linear call-by-push-value L-calculus (without sums) in [CFMM 2016; Munch-Maccagnoni 2017] factors into the previous construction of a symmetric monoidal closed duploid from a linear effect adjunction (thm. 6.5), and the direct interpretation of the calculus into the symmetric monoidal closed duploid.*

The result can be extended to the definition of non-focused constructors, in which case the definition of $V \cdot e$ matches the construction of $\multimap$ as a right adjoint using prop. 6.4.

8 Dialogue duploids

In this section, we will describe the structure inherited by a duploid associated to a dialogue chirality [Melliès 2016] described in the introduction (def. 1.2) as a rephrasing of continuation models. A dialogue duploid is going to be a duploid given with positive and negative symmetric monoidal structures $(\otimes, 1)$ and $(⅋, \bot)$ related by a *duality functor* $(-)^*$. The intuition is that a classical sequent $A_1, \dots, A_n \vdash B_1, \dots, B_m$ will be interpreted by a duploid morphism $A_1 \otimes \cdots \otimes A_n \to B_1 ⅋ \cdots ⅋ B_m$ and $(-)^*$ will interpret negation.

Before formally defining the notion of dialogue duploid, we find convenient to define a notion of strong monoidal functor between symmetric monoidal duploids:

Definition 8.1. A strong monoidal functor

$$F \quad : \quad (\mathcal{D}, \otimes, 1) \longrightarrow (\mathcal{E}, \otimes, 1)$$

between symmetric monoidal duploids is a duploid functor from $\mathcal{D}$ to $\mathcal{E}$ equipped with a family of thunkable and linear isomorphisms

$$\begin{array}{rcrcl} m_{X,Y} & : & FX \otimes FY & \longrightarrow & F(X \otimes Y) \\ m_1 & : & 1 & \longrightarrow & F(1) \end{array}$$

natural in each component X and Y independently, and making the same coherence diagrams commute as in the usual case of a strong monoidal functor between symmetric monoidal categories.

Definition 8.2. A pair of strong monoidal functors $F : \mathcal{D} \to \mathcal{E}$ and $G : \mathcal{E} \to \mathcal{D}$ between symmetric monoidal duploids $(\mathcal{D}, \otimes, 1)$ and $(\mathcal{E}, \otimes, 1)$ is called a **monoidal equivalence** when there exists two families of thunkable and linear isomorphisms $\nu_X : F(GX) \to X$ and $\nu'_X : G(FX) \to X$, both natural in X and compatible with the respective $m_{X,Y}$ and m_1.

This leads us to the following definition of a dialogue duploid.

Definition 8.3. A **dialogue duploid** is a duploid $\mathcal{D}$ equipped with a positive and negative symmetric monoidal duploid structure $(\mathcal{D}, \otimes, 1)$ and $(\mathcal{D}, \parr, \bot)$ related by a strong monoidal equivalence

$$(\mathcal{D}, \otimes, 1) \underset{(-)^*}{\overset{(-)^*}{\rightleftarrows}} (\mathcal{D}, \parr, \bot)^{\mathrm{op}}$$

together with a family of adjunctions $- \otimes Y \vdash Y^* \parr -$ between graph morphisms (called currification)

$$\chi_{X,Y,Z} \quad : \quad \mathcal{D}(X \otimes Y, Z) \quad \xrightarrow{\cong} \quad \mathcal{D}(X, Y^* \parr Z)$$

natural component-wise in X, Y and Z, and subject up to monoidality, symmetry and associativity to the coherence condition between χ and monoidality $\chi_{A,B\otimes C,D} = \chi_{A,B,C^* \parr D} \circ \chi_{A \otimes B,C,D}$.

Note that an associative dialogue duploid is the same thing as a $*$-autonomous category. The theorem below establishes in what sense the notion of dialogue duploid can be seen as a direct and computational counterpart to dialogue chiralities, which provides an overarching mathematical framework for reasoning in direct style about (linear and non-linear continuations), while preserving the perfect symmetry between CBV and CBN evaluation paradigms.

Theorem 8.4. *Every duploid* $\mathbf{dupl}_{L,R}$ *associated to a dialogue chirality* $L \dashv R$ *comes equipped with a dialogue duploid structure. Conversely, every dialogue duploid* $\mathcal{D}$ *induces a dialogue chirality structure on the adjunction* (14)*, whose associated dialogue duploid is equivalent to* $\mathcal{D}$ *via strong monoidal duploid functors that also preserve the duality.*

9 The linear classical L-calculus

In this section, we expand the calculus of section 7 with a disjunction and an involutive negation, whose constructors and rules are presented in fig. 2. We call this expanded calculus the linear classical L-calculus.

Syntax. From the types of the linear call-by-push-value L-calculus, we remove $\multimap$ and add the disjunction $\parr$ as well as its the corresponding unit $\bot$. Furthermore, we split negation $(-)^*$ into two type constructors, one for each polarity (as in [Danos, Joinet, and Schellinx 1997; Munch-Maccagnoni 2014a]). We write the two negation connectives the same, for simplicity and due to lack of ambiguity.

$$\text{Types } A, B, A_\varepsilon ::= \underbrace{X^+ \mid 1 \mid A \otimes B \mid N^*}_{\text{positive } (P,Q,A_+)} \mid \underbrace{X^- \mid \bot \mid A \parr B \mid P^*}_{\text{negative } (N,M,A_-)}$$

Corresponding to the unit $\bot$ and to the disjunction $\parr$, dualizing the unit 1 and the conjunction $\otimes$, we have the nullary and binary binders $\mu[].c$ and $\mu(\alpha \parr \beta).c$ and the constructors $[]$ and $S \parr S'$.

The grammar, conversions and typing rules are those from fig. 1, with the constructors $\mu(a{\cdot}\beta).c$ and $V{\cdot}S$ removed and the following added:

$$V, W ::= \ldots \mid \mu[].c \mid \mu(\alpha \parr \beta).c \mid [S] \mid \mu[a].c$$
$$S ::= \ldots \mid [] \mid S \parr S' \mid \tilde{\mu}[\alpha].c \mid [V]$$

(a) Additional Grammar

$$\begin{array}{llll}
(R\bot) & \langle \mu[].c \parallel [] \rangle^- & \triangleright_R & c \\
(R\parr) & \langle \mu(\alpha\parr\beta).c \parallel S\parr S' \rangle^- & \triangleright_R & c[S/\alpha, S'/\beta] \\
(R-^*) & \langle [S] \parallel \tilde{\mu}[\alpha].c \rangle^+ & \triangleright_R & c[S/\alpha] \\
(R+^*) & \langle \mu[a].c \parallel [V] \rangle^- & \triangleright_R & c[V/a]
\end{array}
\qquad
\begin{array}{llll}
(E\bot) & V & \triangleright_E & \mu[].\langle V \parallel [] \rangle^- \\
(E\parr) & V & \triangleright_E & \mu(\alpha\parr\beta).\langle V \parallel \alpha\parr\beta \rangle^- \\
(E-^*) & S & \triangleright_E & \tilde{\mu}[\alpha].\langle [\alpha] \parallel S \rangle^+ \\
(E+^*) & V & \triangleright_E & \mu[a].\langle V \parallel [a] \rangle^-
\end{array}$$

(b) Additional conversions (Reduction and expansion rules)

$$\frac{}{\mid [] : \bot \vdash}\ (\bot \vdash) \qquad \frac{c : (\Gamma \vdash \Delta)}{\Gamma \vdash \mu[].c : \bot \mid \Delta}\ (\vdash \bot)$$

$$\frac{\Gamma \mid S : A \vdash \Delta \qquad \Gamma' \mid S' : B \vdash \Delta'}{\Gamma, \Gamma' \mid S\parr S' : A \parr B \vdash \Delta, \Delta'}\ (\parr \vdash_f) \qquad \frac{c : (\Gamma \vdash \alpha : A, \beta : B, \Delta)}{\Gamma \vdash \mu(\alpha\parr\beta).c : A \parr B \mid \Delta}\ (\vdash \parr)$$

$$\frac{\Gamma \mid S : N \vdash \Delta}{\Gamma \vdash [S] : N^* \mid \Delta}\ (\vdash_f -^*) \qquad \frac{c : (\Gamma \vdash \alpha : N, \Delta)}{\Gamma \mid \tilde{\mu}[\alpha].c : N^* \vdash \Delta}\ (-^* \vdash)$$

$$\frac{\Gamma \vdash V : P \mid \Delta}{\Gamma \mid [V] : P^* \vdash \Delta}\ (+^* \vdash_f) \qquad \frac{c : (a : P, \Gamma \vdash \Delta)}{\Gamma \vdash \mu[a].c : P^* \mid \Delta}\ (\vdash +^*)$$

(c) Additional typing rules

Fig. 2. Syntax of the linear classical L-calculus

Constructors for an involutive negation. In order to model the rules of negation, we also have the unary binders $\mu[a].c$ and $\tilde{\mu}[\alpha].c$, as well as the constructions $[V]$ and $[S]$ which turn terms into duals. The computational intuition for $|[V] : P^* \vdash$ is that of a stack with a single positive argument and *no return*, where P^* is a negative type of continuations expecting P. On the other hand, the intuition for $\vdash [S] : N^*|$ is that of a value denoting a stack that has been captured (by a control operator) where N^* is a positive type of (inspectable) captured stacks [Levy 2004; Munch-Maccagnoni 2014a].

The rules for negation can create right-hand side contexts Δ with strictly more or less than one formula. The underlying logic of typing judgments is not intuitionistic anymore and is a polarised multiplicative linear logic.

The closure in the linear L-calculus. In the linear classical L-calculus, the closure is definable as:

$$A \multimap B = A^* \parr B$$

Moreover, the rules associated with it presented in fig. 1 can be deduced from the rules for the disjunction and the negation. This is why we do not include $\multimap$ in the definition of the linear classical L-calculus.

Recovering non-focused rules. Recovering the non-focused rules $(\vdash -^*)$ and $(+^* \vdash)$ is a crucial and subtle part about the involutive negation.

Definition 9.1. For e a negative context, we define $[e] := \mu\alpha^{+}.\langle\mu\beta^{-}.\langle[\beta] \parallel \alpha\rangle^{+} \parallel e\rangle^{-}$. Symmetrically, for t a positive term, we define $[t] := \tilde{\mu}a^{-}.\langle t \parallel \tilde{\mu}b^{+}.\langle a \parallel [b]\rangle^{-}\rangle^{+}$. The following rules can be derived:

$$\frac{\Gamma \mid e : N \vdash \Delta}{\Gamma \vdash [e] : N^{*} \mid \Delta}\ (\vdash -^{*}) \qquad \frac{\Gamma \vdash t : P \mid \Delta}{\Gamma \mid [t] : P^{*} \vdash \Delta}\ (+^{*} \vdash)$$

In words, the reduction of $[e]$ and $[t]$ proceeds inside the terms using what looks like let-expansions:

$$\langle [e] \parallel S\rangle^{+} \triangleright_{R} \langle\mu\beta^{-}.\langle[\beta] \parallel S\rangle^{+} \parallel e\rangle^{-} \qquad (e \text{ not a stack})$$

$$\langle V \parallel [t]\rangle^{-} \triangleright_{R} \langle t \parallel \tilde{\mu}b^{+}.\langle V \parallel [b]\rangle^{-}\rangle^{+} \qquad (t \text{ not a value})$$

Notice that these expansions involve cuts of both polarities in the right-hand side. It is therefore not possible to obtain this computational behaviour in calculi that are globally call-by-value or call-by-name, in which negation is a suspension (*e.g.* [Curien and Herbelin 2000; Wadler 2003; Laurent 2011] among others).

A straightforward but nevertheless useful property states that the involutive negation internalises the duality between expressions and contexts:

Lemma 9.2. *For* e *a context and* c *a command, one has* $\langle [e] \parallel \tilde{\mu}[\alpha].\mathrm{c}\rangle^{+} \simeq_{RE} \langle\mu\alpha^{-}.\mathrm{c} \parallel e\rangle^{-}$. *Likewise, for* t *an expression and* c *a command, one has:* $\langle\mu[a].\mathrm{c} \parallel [t]\rangle^{-} \simeq_{RE} \langle t \parallel \tilde{\mu}a^{+}.\mathrm{c}\rangle^{+}$.

Proof. Let e be a negative context and c a command.

$$\begin{aligned}
&\left\langle \underline{\mu\beta^{+}}.\langle\mu\gamma^{-}.\langle[\gamma] \parallel \beta\rangle^{+} \parallel e\rangle^{-} \,\middle\|\, \tilde{\mu}[\alpha].\mathrm{c}\right\rangle^{+} \\
&\simeq_{RE} \langle\mu\gamma^{-}.\langle[\gamma] \parallel \underline{\tilde{\mu}[\alpha]}.\mathrm{c}\rangle^{+} \parallel e\rangle^{-} && (R\mu^{+}) \\
&\simeq_{RE} \langle\mu\gamma^{-}.\mathrm{c}[\gamma/\alpha] \parallel e\rangle^{-} && (R-^{*}) \\
&\simeq_{RE} \langle\mu\alpha^{-}.\mathrm{c} \parallel e\rangle^{-}
\end{aligned}$$

The case of $e = S$ is straightforward. The other case is similar. □

Soundness of the calculus. The results for the linear call-by-push-value L-calculus extend to the linear classical L-calculus. The reader will find detailed proofs in the Appendix I.

Theorem 9.3 (Subject reduction). *If* $\mathrm{c} \rightarrow_{RE} \mathrm{c}'$ *and* $\mathrm{c} : (\Gamma \vdash \Delta)$, *then* $\mathrm{c}' : (\Gamma \vdash \Delta)$.

Theorem 9.4 (Soundness of the linear classical L-calculus). *Given some assigment* ρ *of atoms to objects respecting the polarity, the interpretation of typed terms of the linear classical* L*-calculus in any dialogue duploid, sending sequents* $A_1, \ldots, A_n \vdash B_1, \ldots, B_m$ *to hom-sets* $\mathcal{D}(A_1^{\rho} \otimes \cdots \otimes A_n^{\rho}, B_1^{\rho} \parr \cdots \parr B_m^{\rho})$, *is invariant modulo typed reductions and expansions.*

10 The syntactic dialogue duploid

We now construct a dialogue duploid whose objects are the types of the (two-sided) linear classical L-calculus and whose morphisms $\mathrm{c} : A \rightarrow B$ between two types A and B are the commands $\mathrm{c} : (a : A \vdash \beta : B)$ quotiented by the rewriting relation $\simeq_{RE}$. The composite of two maps

$$\mathrm{c} : (a : A \vdash \beta : B_{\varepsilon}) \qquad \mathrm{c}' : (b : B \vdash \gamma : C)$$

with respective typing derivations π_1 and π_2, is defined as the command of the L-calculus:

$$\langle\mu\beta^{\varepsilon}.\mathrm{c} \parallel \tilde{\mu}b^{\varepsilon}.\mathrm{c}'\rangle^{\varepsilon}$$

with typing derivation:

$$\dfrac{\dfrac{\dfrac{\pi_2}{\mathrm{c}' : (b : B \vdash \gamma : C)}}{\mid \tilde{\mu} b^{\varepsilon}.\mathrm{c}' : B \vdash \gamma : C}\,(\tilde{\mu}\vdash) \qquad \dfrac{\dfrac{\pi_1}{\mathrm{c} : (a : A \vdash \beta : B)}}{a : A \vdash \mu\beta^{\varepsilon}.\mathrm{c} : B \mid}\,(\vdash \mu)}{\langle \mu\beta^{\varepsilon}.\mathrm{c} \parallel \tilde{\mu} b^{\varepsilon}.\mathrm{c}' \rangle^{\varepsilon} : (a : A \vdash \gamma : C)}\,(\mathrm{cut})$$

Theorem 10.1. *The construction just described defines a dialogue duploid called the syntactic dialogue duploid.*

In order to establish the theorem, we give the following characterizations of thunkable maps and of central maps in the non-associative category of commands. Linear maps are characterized symmetrically.

Lemma 10.2 (Adapted from [CFMM 2016]). *Let $a : A \vdash t : B \mid$ be an expression. The two following properties are equivalent :*

(1) *For all commands* $\mathrm{c} : (b : B_{\varepsilon} \vdash \gamma : C)$, $\langle t \parallel \tilde{\mu} b^{\varepsilon}.\mathrm{c} \rangle^{\varepsilon} \simeq_{RE} \mathrm{c}[t/b]$;
(2) *For all commands* $\mathrm{c} : (b : B_{\varepsilon} \vdash \gamma : C_{\varepsilon'})$ *and contexts* $\mid e : C_{\varepsilon'} \vdash \delta : D$,

$$\langle t \parallel \tilde{\mu} b^{\varepsilon}.\langle \mu\gamma^{\varepsilon'}.c \parallel e \rangle^{\varepsilon'} \rangle^{\varepsilon} \simeq_{RE} \langle \mu\gamma^{\varepsilon'}.\langle t \parallel \tilde{\mu} b^{\varepsilon}.c \rangle^{\varepsilon} \parallel e \rangle^{\varepsilon'} .$$

We say that an expression t is **syntactically thunkable** *when it satisfies one of the above equivalent properties.*

Lemma 10.3. *A command* $\mathrm{c} : (a : A \vdash \beta : B)$ *is thunkable if and only if* $\mu\beta^{\varepsilon_B}.\mathrm{c}$ *is syntactically thunkable.*

This characterization based on the intuition that thunkable expression behave like values plays a fundamental role in the proof that the syntactic polarity ε of a type A_{ε} in the L-calculus coincides with its semantic polarity as an object of the non-assocative category, as it is defined in def. 3.1.

Definition 10.4. An expression t is **syntactically central** when the equality up to reduction and expansion is satisfied

$$\langle t \parallel \tilde{\mu} q_1.\langle u \parallel \tilde{\mu} q_2.c \rangle^{\varepsilon_2} \rangle^{\varepsilon_1} \simeq_{RE} \langle u \parallel \tilde{\mu} q_2.\langle t \parallel \tilde{\mu} q_1.c \rangle^{\varepsilon_1} \rangle^{\varepsilon_2}$$

for all commands c, expressions u and binders q_1 and q_2 (i.e. either a, $a \otimes b$, $()$ or $[\alpha]$) of polarity ε_1 and ε_2 respectively.

Lemma 10.5. *A command* $\mathrm{c} : (a : A \vdash \beta : B)$ *is central if and only if the expression* $\mu\beta^{\varepsilon_B}.\mathrm{c}$ *is syntactically central.*

This characterization of central commands in the L-calculus is the basis of the proof that (positive) thunkable commands are central, and thus, that the tensor $\otimes$ of the syntax defines a positive monoidal structure. The interested reader will find the proofs of the two lemmas in the Appendix.

11 The Hasegawa-Thielecke theorem

In this section, we formulate and establish the Hasegawa-Thielecke theorem in the language of dialogue duploids. We have seen in prop. 5.1 that every thunkable map is central in a symmetric monoidal duploid, and that the converse property is not true in general. We establish now that the two notions coincide in a dialogue duploid.

Theorem 11.1 (Hasegawa-Thielecke). *In a dialogue duploid, a morphism is central for $\otimes$ if and only if it is thunkable.*

A direct proof by equational reasoning is given in Appendix K. It relies on the crucial observation that the composite $g \circledcirc f$ for every pair of maps $f : A \to B$ and $g : B \to C$ can be expressed in every dialogue duploid by internal duality:

$$g \circledcirc f = \varphi_C^{-1}(\varphi_B(f) \bullet (A \rtimes g^*)) \quad : \quad A \to C$$

where $\varphi_D : \mathcal{D}(A, D) \xrightarrow{\cong} \mathcal{D}(A \otimes D^*, \bot)$ is obtained from $\chi^{-1}_{A,D^*,\bot}$, the duality $D^{**} \cong D$, and unitors. The proof derives in fact from our earlier observation (§1.5) that the two distinct commutation properties characterising thunkability (9) and centrality (10) in sequent calculus, respectively thm. 10.2 and def. 10.4, coincide via duality.

One benefit of the linear classical L-calculus is that the same statement can be also established by syntactic means in sequent calculus, thanks to its equational theory and the soundness theorem. This allows us to make the explanation from §1.5 rigorous with a (brief!) proof of the Hasegawa-Thielecke theorem that directly relates by internal duality the characterisations of thunkability and centrality.

Theorem 11.2 (Syntactic Hasegawa-Thielecke theorem). *An expression of the linear classical L-calculus is syntactically central for $\otimes$ if and only if it is syntactically thunkable.*

Proof. As we have already seen, an expression which is syntactically thunkable is also syntactically central. Now assume that t is syntactically central. To prove that t is syntactically thunkable, let c be a command and e be a context as per thm. 10.2. We need to establish

$$\langle t \parallel \tilde{\mu}b.\langle \mu\gamma.\mathrm{c} \parallel e\rangle\rangle \simeq_{RE} \langle \mu\gamma.\langle t \parallel \tilde{\mu}b.\mathrm{c}\rangle \parallel e\rangle$$

The only difficult case is when t is positive and e is negative. Using internal duality (lem. 9.2) twice, one indeed has:

$$\begin{aligned}
&\langle t \parallel \tilde{\mu}b^+.\langle \mu\gamma^-.\mathrm{c} \parallel e\rangle^-\rangle^+ \\
&\simeq_{RE} \langle t \parallel \tilde{\mu}b^+.\langle [e] \parallel \tilde{\mu}[\gamma].\mathrm{c}\rangle^+\rangle^+ && \text{by lem. 9.2}\\
&\simeq_{RE} \langle [e] \parallel \tilde{\mu}[\gamma].\langle t \parallel \tilde{\mu}b^+.\mathrm{c}\rangle^+\rangle^+ && \text{by centrality of } t\\
&\simeq_{RE} \langle \mu\gamma^-.\langle t \parallel \tilde{\mu}b^+.\mathrm{c}\rangle^+ \parallel e\rangle^- && \text{by lem. 9.2}
\end{aligned}$$

□

Now recall that the general situation of a duploid $\mathcal{D}$ associated to an adjunction $L \dashv R$, one has that

> *the monad $R \circ L$ is idempotent if and only if every morphism of the duploid $\mathcal{D}$ is thunkable.*

Also, it is not difficult to see that in the situation described in §4 of a symmetric monoidal duploid $\mathcal{D}$ associated to an adjunction $L \dashv R$ where $\mathscr{A}$ is symmetric monoidal and where the monad $T = R \circ L$ is strong, one has that

> *the monad T is commutative if and only if every morphism of the duploid $\mathcal{D}$ is central.*

In the case of a dialogue duploid $\mathcal{D}$ associated to a dialogue category, this proves as a corollary of thm. 11.2 the following statement, attributed to Hasegawa in Melliès and Tabareau [2010].

Corollary 11.3. *The continuation monad of a dialogue category is commutative if and only if it is idempotent.*

It is natural to wonder if we could not weaken the assumptions of structure on duploids. Removing negation from fig. 2 leads to consider a linearly distributive structure on duploids:

Definition 11.4. A **linearly distributive duploid** is a duploid equipped with a pair of positive and negative symmetric monoidal structures related by a family of mappings $A \otimes (B \parr C) \to (A \otimes B) \parr C$ natural component-wise and that respects the usual coherence diagrams for a linearly distributive category [Cockett and Seely 1997; Melliès 2017a]. (Note in particular that a linearly distributive duploid that is associative is the same thing as a linearly distributive category.)

A variant of the syntactic argument given in Munch-Maccagnoni [2013, p.262] then suggests the following refinement of the Hasegawa-Thielecke theorem (in the dual): in any linearly distributive duploid which is closed (in the sense of an isomorphism $\mathcal{D}(X \otimes Y, Y' \parr Z) \cong \mathcal{D}(X, (Y \multimap Y') \parr Z)$ natural in X, Y', Z component-wise), a morphism is central for $\parr$ if and only if it is linear.

12 Variant: the one-sided classical L-calculus

It is possible to present a simplified linear classical L-calculus with all formulae in negative normal form and placed on the right-hand side of sequents, as in the original presentations of linear logic and the polarised classical logic **LC** [Girard 1987, 1991]. This is the presentation retained in the third author's one-sided classical L-calculus [Munch-Maccagnoni 2009].

Formulae being in negative normal form means that negation is no longer a connective except in front of atoms. Negation is now an operation on formulae, written $\overline{\cdot}$, defined by De Morgan laws:

$$\overline{A \otimes B} \overset{\text{def}}{=} \overline{A} \parr \overline{B} \qquad \overline{1} \overset{\text{def}}{=} \bot \qquad \overline{\overline{X}} \overset{\text{def}}{=} X$$
$$\overline{A \parr B} \overset{\text{def}}{=} \overline{A} \otimes \overline{B} \qquad \overline{\bot} \overset{\text{def}}{=} 1$$

In particular, negation is strictly involutive rather than only up to type isomorphism:

$$\overline{\overline{A}} = A$$

Sequents are of the form $\vdash \Gamma$ and left-introduction rules then coincide with the right-introduction rules of the dual connective.

One goal of this paper has been to explain **LC**'s involutive negation by giving a categorical and syntactic account where duality is present as an explicit connective. In this section we go further and describe a one-sided calculus as a strictification of the two-sided calculus, by relating this simplification of the syntax to the coherence theorem between dialogue chiralities and dialogue categories presented in Melliès [2016] (via thms. 4.3 and 9.4 which provide an interpretation of the direct syntax into the indirect models). The one-sided calculus internalises the Hasegawa-Thielecke theorem, in the sense that the strictification makes the sequent calculus derivations characterising thunkability (9) and centrality (10) coincide formally.

The one-sided linear classical L-calculus. We present the one-sided linear classical L-calculus in fig. 3. Informally, one can think of a one-sided sequent

$$\vdash \Gamma$$

as any of its two-sided unfoldings, chosen as deemed convenient:

$$\Gamma_1 \vdash \Gamma_2 \quad \text{where} \quad \overline{\Gamma_1}, \Gamma_2 \text{ is a reordering of } \Gamma \tag{17}$$

as obtained by free applications of exchange and the following informal negation rules:

$$\frac{\Gamma \vdash A, \Delta}{\Gamma, \overline{A} \vdash \Delta} \qquad \frac{\Gamma, A \vdash \Delta}{\Gamma \vdash \overline{A}, \Delta}$$

$$\begin{array}{lrcl} \text{Types} & A, B, A_\varepsilon & ::= & P \mid N \\ \text{Positives:} & P, Q, A_+ & ::= & X \mid 1 \mid A \otimes B \\ \text{Negatives:} & N, M, A_- & ::= & \overline{X} \mid \bot \mid A \parr B \end{array}$$

(a) Formulae

$$\begin{array}{lrcl} \text{(Co)Values:} & V, W & ::= & x, y, \ldots \mid \mu x^-.\mathrm{c} \mid () \mid V \otimes W \mid \mu().\mathrm{c} \mid \mu(x \parr y).\mathrm{c} \\ \text{(Co)Expressions:} & t, u & ::= & V \mid \mu x^+.\mathrm{c} \\ \text{Commands:} & \mathrm{c} & ::= & \langle t \parallel V \rangle \end{array}$$

(b) Grammar

$$\begin{array}{llll} (R\mu^+) & \langle \mu x^+.\mathrm{c} \parallel V \rangle \triangleright_R \mathrm{c}[V/x] & (E\mu^+) & t \triangleright_E \mu x^+.\langle t \parallel x \rangle \\ (R\mu^-) & \langle V \parallel \mu x^-.\mathrm{c} \rangle \triangleright_R \mathrm{c}[V/x] & (E\mu^-) & V \triangleright_E \mu x^-.\langle x \parallel V \rangle \\ (R1/\bot) & \langle () \parallel \mu().\mathrm{c} \rangle \triangleright_R \mathrm{c} & (E1/\bot) & V \triangleright_E \mu().\langle () \parallel V \rangle \\ (R\otimes/\parr) & \langle V \otimes W \parallel \mu(x \parr y).\mathrm{c} \rangle \triangleright_R \mathrm{c}[V/x, W/y] & (E\otimes/\parr) & V \triangleright_E \mu(x \parr y).\langle x \otimes y \parallel V \rangle \end{array}$$

(c) Conversions (Reduction and expansion rules)

$$\mathrm{c} : (\vdash \Gamma) \qquad \vdash t : A \mid \Gamma$$

(d) Judgements

$$\frac{}{\vdash x : A \mid x : \overline{A}}\ (\mathbf{ax}) \qquad \frac{\mathrm{c} : (\vdash x : A_\varepsilon, \Gamma)}{\vdash \mu x^\varepsilon.\mathrm{c} : A_\varepsilon \mid \Gamma}\ (\boldsymbol{\mu}^\varepsilon) \qquad \frac{\vdash t : P \mid \Gamma \qquad \vdash V : \overline{P} \mid \Gamma'}{\langle t \parallel V \rangle : (\vdash \Gamma, \Gamma')}\ (\mathbf{cut})$$

$$\forall \sigma \in \Sigma(\Gamma', \Gamma) : \quad \frac{\vdash t : A \mid \Gamma}{\vdash t[\sigma] : A \mid \Gamma'}\ (\vdash \sigma) \qquad \frac{\mathrm{c} : (\vdash \Gamma)}{\mathrm{c}[\sigma] : (\vdash \Gamma)}\ (\sigma)$$

(e) Typing rules (Identity and structural groups)

$$\frac{}{\vdash () : 1 \mid}\ (1) \qquad \frac{\mathrm{c} : (\vdash \Gamma)}{\vdash \mu().\mathrm{c} : 1 \mid \Gamma}\ (\bot)$$

$$\frac{\vdash V : A \mid \Gamma \qquad \vdash W : B \mid \Gamma'}{\vdash V \otimes W : A \otimes B \mid \Gamma, \Gamma'}\ (\otimes) \qquad \frac{\mathrm{c} : (\vdash x : A, y : B, \Gamma)}{\vdash \mu(x \parr y).\mathrm{c} : A \parr B \mid \Gamma}\ (\parr)$$

(f) Typing rules (Logic group)

Fig. 3. Syntax of the one-sided classical L-calculus

Similarly, the one-sided calculus can be thought of as the two-sided calculus where the term formers of negation are omitted:

$$\frac{\Gamma \vdash t : A \mid \Delta}{\Gamma \mid t : \overline{A} \vdash \Delta} \qquad \frac{\Gamma \mid t : A \vdash \Delta}{\Gamma \vdash t : \overline{A} \mid \Delta} \qquad \frac{c : (\Gamma \vdash x : A, \Delta)}{c : (\Gamma, x : \overline{A} \vdash \Delta)} \qquad \frac{c : (\Gamma, x : A \vdash \Delta)}{c : (\Gamma \vdash x : \overline{A}, \Delta)} \tag{18}$$

As a matter of fact, if we introduce the following notations:

$$\langle V \parallel t \rangle^- \stackrel{\text{def}}{=} \langle t \parallel V \rangle^+ \stackrel{\text{def}}{=} \langle t \parallel V \rangle$$

then we are free to use whichever notation is more convenient regarding the computational interpretation of sequents.

For instance, as is well known, the call-by-name $\lambda\mu$-calculus can be recovered in the negative fragment of polarised classical logic. The Krivine abstract machine is recovered in the one-sided classical L-calculus with the previous notation and the following ones:

$$\lambda x.V \stackrel{\text{def}}{=} \mu(x \parr y).\langle V \parallel y\rangle^- \qquad\qquad V\,W \stackrel{\text{def}}{=} \mu x^-.\langle V \parallel W \otimes x\rangle^-$$

Indeed, one has with these notations:

$$\langle V\,W \parallel S\rangle \rhd_R \langle V \parallel W \otimes S\rangle \qquad\qquad \langle \lambda x.V \parallel W \otimes S\rangle^- \rhd_R \langle V[W/x] \parallel S\rangle^-$$

which correspond to the rules of the Krivine abstract machine, where V, W denote negative expressions (which are indeed values) and S a stack: a positive value of the form $V_1 \otimes \cdots \otimes V_n \otimes x$ where x is a (co)variable denoting the end of the stack.

It is a nice exercise to reproduce using similar notations the sequent calculus derivations characterising thunkability (9) and centrality (10), as this shows that they indeed coincide in the one-sided calculus.

Correspondence with the two-sided L-calculus. Formally, terms from the two-sided L-calculus correspond to terms from the one-sided L-calculus via a surjective mapping $\underline{\cdot}$ from the latter to the former, which we call *folding*, obtained by erasing term formers of negation:

$$\begin{aligned}
\underline{\langle t \parallel S\rangle^+} &= \langle \underline{t} \parallel \underline{S}\rangle & \underline{\langle V \parallel e\rangle^-} &= \langle \underline{e} \parallel \underline{V}\rangle\\
\underline{\mu\alpha^+.c} &= \mu\underline{\alpha}^+.\underline{c} & \underline{\tilde\mu x^-.c} &= \mu x^+.\underline{c}\\
\underline{\mu\alpha^-.c} &= \mu\underline{\alpha}^-.\underline{c} & \underline{\tilde\mu x^+.c} &= \mu x^-.\underline{c}\\
\underline{()} &= () & \underline{[]} &= ()\\
\underline{V \otimes W} &= \underline{V} \otimes \underline{W} & \underline{S \parr S'} &= \underline{S} \otimes \underline{S'}\\
\underline{\mu[].c} &= \mu().\underline{c} & \underline{\tilde\mu().c} &= \mu().\underline{c}\\
\underline{\mu(\alpha \parr \beta).c} &= \mu(\underline{\alpha} \parr \underline{\beta}).\underline{c} & \underline{\tilde\mu(x \otimes y).c} &= \mu(x \parr y).\underline{c}\\
\underline{[S]} &= \underline{S} & \underline{[V]} &= \underline{V}\\
\underline{\mu[x].c} &= \mu x^-.\underline{c} & \underline{\tilde\mu[\alpha].c} &= \mu\underline{\alpha}^-.\underline{c}
\end{aligned}$$

Folding moreover sends the reduction and expansion rules into corresponding reduction and expansion rules, except for the rules $R\pm^*$ and $E\pm^*$ which are sent into the equality of terms. Noticing that those reductions and expansions related to negation are linear in their metavariables and therefore do not affect computation, folding preserves many computational properties of the term. (For instance, it is easy to see that an untyped command c is strongly normalising if and only if $\underline{c}$ is strongly normalising.)

This relates to the biequivalence between (2-categories of) dialogue categories and chiralities [Melliès 2016] in the following way. It so happens that continuation semantics are themselves "one-sided", in the sense that they do not distinguish between the two-sided and the one-sided calculi. This is first reflected in the fact that a dialogue category

$$\mathscr{C} \underset{\neg}{\overset{\neg}{\underset{\bot}{\rightleftarrows}}} \mathscr{C}^{\mathrm{op}}$$

seen as a dialogue chirality

$$(\mathcal{A}, \owedge, \mathit{true}) \quad \underset{(-)^*}{\overset{(-)^*}{\simeq}} \quad (\mathcal{B}, \ovee, \mathit{false})^{\mathrm{op}}$$

that is, with

$$(\mathcal{A}, \owedge, \mathit{true}) = (\mathscr{C}, \otimes, 1) \qquad (\mathcal{B}, \ovee, \mathit{false}) = (\mathscr{C}^{\mathrm{op}}, \otimes, 1)\ ,$$

where dualities $(-)^*$ are given by the identity functors on $\mathscr{C}$ and $\mathscr{C}^{\mathrm{op}}$, and where negation $\neg$ is seen as a pair of (covariant) functors $L : \mathcal{A} \to \mathcal{B}$ and $R : \mathcal{B} \to \mathcal{A}$, is indeed a dialogue chirality that (trivially) statisfies strict identities $A^{**} = A$, $(A \owedge B)^* = A^* \ovee B^*$, etc. (This chirality $(\mathscr{C}, \mathscr{C}^{\mathrm{op}})$ is written $\mathcal{F}\mathscr{C}$ in Melliès [2016].) This lets us refine our interpretation of the two-sided calculus into dialogue chiralities (via dialogue duploids) into an interpretation of the one-sided calculus into dialogue categories.

This furthermore leads to the observation that the interpretation of typed terms of the two-sided L-calculus into any dialogue category $\mathscr{C}$, seen as dialogue chirality $(\mathscr{C}, \mathscr{C}^{\mathrm{op}})$, factors into folding $\underline{\cdot}$ followed by the interpretation of the one-sided L-calculus into dialogue categories; in other words the following diagram always commutes:

$$\begin{array}{ccc} \text{2-sided } L & \xrightarrow{\ \underline{\cdot}\ } & \text{1-sided } L \\ \downarrow & & \downarrow \\ (\mathscr{C}, \mathscr{C}^{\mathrm{op}}) & \xleftarrow[\ \mathcal{F}\ ]{} & \mathscr{C} \end{array}$$

Traditional continuation semantics, by being intrinsically "one-sided" in this way—with every object A being for instance identified with its negation A^* in the opposite category—obscure the algebraic description of the involutive negation of **LC**, which might explain why this involutive negation has at times been misunderstood.

At this point, we find convenient to offer an alternative perspective based on seeing the correspondence between dialogue categories and chiralities as a coherence result [Melliès 2016]. On the semantic side, we observe that dialogue chiralities (and dialogue duploids), better than dialogue categories, provide an accurate and explicit account the involutive negation of **LC** in a polarised setting, and thus,

> *on the semantic side, dialogue chiralities and dialogue duploids offer an explicit and more accurate view on continuations.*

Things are reversed on the syntactic side: the one-sided syntax is worth considering, as it brings a considerable syntactic simplification—if we keep in mind that one-sided sequents stand for arbitrary unfoldings (17) and free applications of the negation rules (18). Thus,

> *on the syntactic side, the one-sided classical L-calculus offers an accurate and more implicit view on continuations.*

13 Classical notions of computations: turning around Joyal's obstruction theorem

André Joyal made the important observation (recalled below, see thm. 13.1) that it is not possible to develop a proof-theoretic account of classical logic using the language of usual (associative) cartesian categories. A simple argument shows that every return object $\bot$ in a symmetric monoidal category $\mathscr{C}$ induces a family of canonical maps

$$\eta_A \quad : \quad A \longrightarrow \neg\neg A \tag{19}$$

indexed by the objects A of the category $\mathscr{C}$, which reflects the logical principle that every formula A implies its double negation $\neg\neg A$. This family of maps is the unit of the self-adjunction of negation with itself, mentioned in (7). A return object $\bot$ is called *dualizing* when the canonical map (19) is an isomorphism for every object A. A natural direction to resolve the quest for a proof-theoretic interpretation of classical logic would be to look for a cartesian category $(\mathscr{C}, \times, 1)$ equipped with a dualizing object $\bot$. Unfortunately, Joyal observed that the search for such a simple solution cannot succeed:

Theorem 13.1 (Joyal's obstruction theorem). *Any cartesian category $(\mathscr{C}, \times, 1)$ with a dualizing object $\bot$ is a preorder, and thus defines a boolean algebra (up to equivalence).*

Proof. See Appendix A. ◻

For a long time, this observation has been widely accepted as evidence that classical logic cannot be interpreted in a denotational and proof-relevant way. The situation changed in the early 1990s when Griffin [1990] and Murthy [1991] observed a fundamental and unexpected relationship between proof systems for classical logic, and programs written with the control operator C, a variant of Scheme's *call-cc*. Since then, a large number of investigations have been made to define a clean denotational and proof-theoretic interpretation of classical logic. Interestingly, each of the two main directions taken can be seen as providing a specific way to relax one of the hypothesis of Joyal's obstruction theorem:

1) *Classical linear logic [Girard 1987]:* the idea is to relax the cartesianity condition and to work with $*$-autonomous categories, defined as symmetric monoidal categories $(\mathscr{C}, \otimes, 1)$ equipped with a dualizing object $\bot$, possibly supplemented with an exponential modality $A \mapsto\ !A$ to deal with non-linearity,

2) *Continuation models:* the idea is to relax the dualizing condition, and work with categories where (19) has a section or a retraction, obtained from continuation-passing style (CPS) constructions over cartesian categories $(\mathscr{C}, \times, 1)$ equipped with a return object $\bot$.

In these two directions, influential and most notable works have been the Lafont-Reus-Streicher translation [Lafont, Reus, and Streicher 1993] as well as the later works by Hofmann and Streicher [2002] and by Selinger [2001]. Another important and early work has been the introduction of two dual sequent calculi **LKT** and **LKQ** for classical logic, and their translation in linear logic by Danos, Joinet, and Schellinx [1993, 1995, 1997], which turned out to rephrase respectively the CBV and CBN CPS semantics [Ogata 2000]. Interestingly, all these models "break the symmetry" of classical logic by giving precedence at some stage to the CBV or CBN side. The symmetry between the two sides remains however, as a categorical duality observed by Streicher and Reus [1998] and made manifest by Selinger [2001] and Curien and Herbelin [2000] (predated by, and in the spirit of, Filinski's *symmetric λ-calculus* [Filinski 1989]).

Curien and Herbelin's original L-calculus explored in particular a syntactic symmetry between the CBN and CBV calculi which reflects the categorical duality. It was discovered through the reunion of two research lines—the one we just mentioned around the connection between constructive classical logic and CPS [Danos, Joinet, and Schellinx 1995, 1997; Ogata 2000], and the one that investigated well-behaved λ-calculi for classical logic [Parigot 1992] and sequent calculus [Herbelin 1994]. Curien and Herbelin's L-calculus was also inspired by Barbanera and Bernardi's *symmetric λ-calculus* [Barbanera and Berardi 1996] whose reduction is non-deterministic. In a similar line of research, an order-enriched categorical interpretation of classical logic [Führmann and Pym

2006] based on a non-deterministic calculus [Urban 2000] was described in [Bellin et al. 2006]. The connections between this line of research based on a non-deterministic interpretation of cut elimination, and the present paper based on the polarised and deterministic linear classical L-calculus, remain to be clarified.

3) *Preserving the symmetries of classical logic at the expense of associativity:* at about the same time as Griffin [1990] and Murthy [1991], in the early 1990s, an elegant and third direction, inspired by linear logic, was explored by Girard [1991] with the classical logic **LC**. The goal was to preserve the symmetries of logic—in particular, an involutive negation and various De Morgan identities present as type isomorphisms—by giving a formal status to the notion of polarity of a formula. Girard's work on **LC** inspired many later works (Murthy [1992], Quatrini and Tortora De Falco [1996], Laurent [2002], Zeilberger [2008], Liang and Miller [2009], and Melliès and Tabareau [2010] among others) including in fact some of the works we already mentioned [Lafont, Reus, and Streicher 1993; Danos, Joinet, and Schellinx 1995, 1997].

The solution, which involves giving up the associativity of composition precisely in the way which we have described, had not seen much exploration from the angle of categorical proof theory. In fact, the question of categorical proof for classical logic theory was essentially mentioned as open in Hyland [2002]. This is the direction we took in the present paper. By applying the duploid construction to dialogue categories with linear classical logic in mind, we showed that the approach from Girard's **LC** makes sense from a semantic point of view with dialogue duploids, related to the syntactic point of view of the L-calculus.

14 Conclusion and future work

We have introduced the syntax and semantics of linear L-calculus, and developed theories of symmetric monoidal duploids with closure and with involutive negation. We see the framework as a solid foundation for the study of non-associative and effectful logical systems and term calculi for linearity, effects, and classical logic, integrating the lessons of linear logic, continuation models and functorial game semantics. The perspective of an encompassing theory of non-associative direct models for effectful programs and proofs is promising, but much remains to be done in this respect.

One interesting application area for such a classical calculus is the study of subtle design issues in continuations for programming languages. For instance, Cong et al. [2019] characterise a restriction to the usage of continuations suitable for compilation, which is not as strong as linearity: crucially, its still permits to copy and discard continuations. This is beyond the scope of dialogue duploids and could involve the notion of linearly distributive duploid just introduced.

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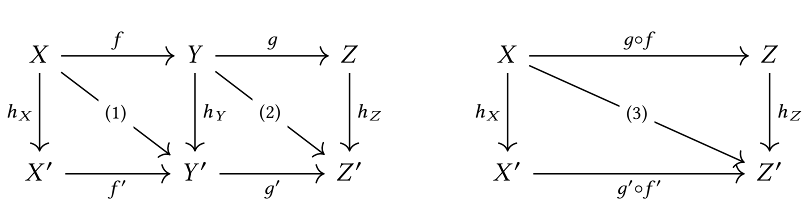
A A proof of Joyal's obstruction theorem

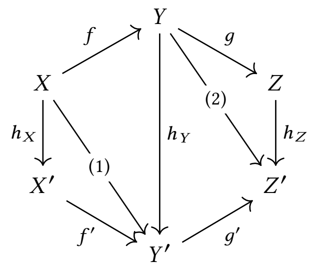
Proof of Joyal's obstruction theorem. Let $(\mathscr{C}, \times, 1)$ a cartesian category with a dualizing object $\bot$. One has natural bijections $\mathscr{C}(A, \bot) \cong \mathscr{C}(A \times 1, \bot) \cong \mathscr{C}(A, \bot^1)$ hence $\bot \cong \bot^1$. Observe then that the object $\bot^\bot \cong \bot^{\bot^1} \cong 1$ is the terminal object. Consequently, the set $\mathscr{C}(\bot \times \bot, \bot)$, in bijection with $\mathscr{C}(\bot, \bot^\bot)$, is a singleton; in particular one has $\pi_1 = \pi_2 \in \mathscr{C}(\bot \times \bot, \bot)$. Now consider the pairs $\langle f, g\rangle \in \mathscr{C}(A, \bot \times \bot)$ for $f, g \in \mathscr{C}(A, \bot)$. By the identity of projections, one has $f = g$ for any such pair of morphisms, in other words any $\mathscr{C}(A, \bot)$ has at most one element. Thus, any hom-set $\mathscr{C}(B, C)$ has at most one element as well, as witnessed by the bijections: $\mathscr{C}(B, C) \cong \mathscr{C}(B, \bot^{\bot^C}) \cong \mathscr{C}(B \times \bot^C, \bot)$. □

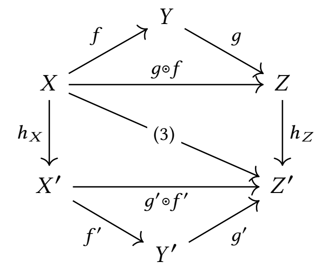
B Chasing and rewriting triangulated commutative diagrams

As we mentioned, very basic principles of usual associative categories are not necessarily true anymore in non-associative categories. In particular, the fact that the triangulated diagram on the left commutes in the sense that $h_Y \circledcirc f = f' \circledcirc h_X$ and $h_Z \circledcirc g = g' \circledcirc h_Y$



does not imply that the diagram (3) commutes in the sense that $h_Z \circledcirc (g \circledcirc f) = (g' \circledcirc f') \circledcirc h_X$. The reason is that one cannot apply a series of flips to transform the triangular decomposition (a) of the hexagon on the left to the decomposition (b) of the same hexagon on the right:

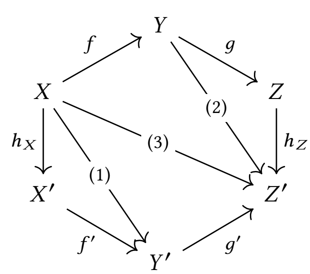
(a) = 

(b) = 

However, the sequence of computations below establishes that diagram (3) commutes when the three paths of length three of the diagram associate:

$$\begin{aligned}
h_Z \circ (g \circ f) &= (h_Z \circ g) \circ f && \text{the path } (f, g, h_z) \text{ associates} && (20)\\
&= (g' \circ h_Y) \circ f && \text{equation (2)} && (21)\\
&= g' \circ (h_Y \circ f) && \text{the path } (f, h_Y, g') \text{ associates} && (22)\\
&= g' \circ (f' \circ h_X) && \text{equation (1)} && (23)\\
&= (g' \circ f') \circ h_X && \text{the path } (h_X, f', g') \text{ associates} && (24)
\end{aligned}$$

The equation can be also established diagrammatically by flipping the map h_Y into the map (3) using the fact that the path (f, h_Y, g') associates, in order to obtain the commutative triangulation, from which it is easy to obtain the commutative triangulation (b) by flipping (1) and (2) into $g \odot f$ and $g' \odot f'$ using the fact that the two paths (f, g, h_Z) and (h_X, f', g') associate.

(c) = 

C Non functoriality of the shift operator: an illustration

We consider the duploid associated to the finite distribution monad T already discussed in section 1.3 and explain why the positive shift $X \mapsto \Downarrow X$ is not functorial in that specific example. The positive shift $\Downarrow X$ associated to an object X of the duploid is defined by case analysis on the polarity of X:

- If $X = (0, A)$, then the shifted object $\Downarrow(0, A) := (0, A)$ is equal to the original object.
- If $X = (1, B)$, then the shifted object $\Downarrow(1, B) := (0, TB)$ is the set TB of finite probability distributions of the set B, seen as a positive object.

Since the definition of $\Downarrow X$ depends on the polarity as we have just seen, it is also the case for the definition of the map $\omega_X : X \to \Downarrow X$.

- $\omega_{(0,A)} : (0, A) \to (0, A)$ is the identity,
- $\omega_{(1,B)} : (1, B) \to (0, TB)$ is the map $d \mapsto 1 \mid d\rangle$ which associates to every distribution $d \in TB$ the Dirac distribution of distributions $1 \mid d\rangle \in TTB$.

Now, suppose given a map $f : X \to Y$ in the duploid. In the same way as for $\omega_X : X \to \Downarrow X$, the construction of the linear map $f^\dagger : \Downarrow X \to Y$ depends on the polarity of X.

- If $X = (0, A)$ then the linear map $f^\dagger : (0, A) \to Y$ is defined as $f^\dagger = f$,

- If $X = (1, B)$ then the map $f : (1, B) \to Y$ is the same thing as a stochastic map $f : TB \to Y$ where we identity Y with its underlying set. The linear map $f^\dagger : (0, TB) \to Y$ may be thus defined as the stochastic map underlying f.

Suppose given a map $f : X \to Y$ in the duploid. The definition of the map $\Downarrow f$ depends on the polarity of X and Y. We give the precise description for X and Y both positive and X and Y both negative, as the other cases can be easily deduced.

- When $X = (0, A)$ and $Y = (0, A')$, the map $\Downarrow f : (0, A) \to (0, A')$ is equal to f.
- When $X = (1, B)$ and $Y = (1, B')$, the map f is a stochastic map $f : TB \to B'$ of the general form
$$f = d \mapsto \sum_i p_i(d) \mid f_i(d)\rangle.$$
The map $\Downarrow f : (0, TB) \to (0, TB')$ is the stochastic map $\Downarrow f : TB \to TB'$ which transports every distribution to the Dirac distribution of its image by f:
$$\Downarrow f = d \mapsto 1 \Big| \sum_i p_i(d) \mid f_i(d)\rangle \Big\rangle$$

So, one should think of the positive shift as a *wrap* operation obtained by turning every output distribution $d' \in TB$ into the Dirac distribution $1 \mid d'\rangle \in TTB$. In particular, if we define the two maps f and g as the following maps
$$\begin{array}{c} f : (0, A) \to (0, B) \\ f := a \mapsto \sum_i p_i(a) \mid f_i(a)\rangle \end{array} \qquad \text{and} \qquad \begin{array}{c} g : (0, B) \to (1, B) \\ g : b \mapsto 1 \mid b\rangle \end{array},$$
then, we have
$$\Downarrow(g \circledcirc f) := 1 \Big| \sum_i p_i(a) \mid f_i(a)\rangle \Big\rangle \qquad \text{and} \qquad \Downarrow g \bullet \Downarrow f := \sum_i p_i(x) \Big| 1 \mid f_i(x)\rangle \Big\rangle$$

This implies that $\Downarrow(g \circledcirc f) = \Downarrow g \bullet \Downarrow f$ precisely when f transports every input $x \in A$ to a Dirac distribution. Note that we have established in the introduction (§1.3) that this property characterizes the thunkable maps of the duploid.

D Graph morphisms and adjunctions between them

Definition D.1. Let $\mathcal{E}$ a non-associative category and $F, G : \mathcal{D} \rightharpoonup \mathcal{E}$ two graph morphisms into $\mathcal{E}$. A **natural transformation** τ from F to G is given by, for every object $A \in \mathcal{D}$ a morphism $\tau_A : FA \to GA$, such that for all $f \in \mathcal{D}(A, B)$ one has $\tau_B \circledcirc Ff = Gf \circledcirc \tau A$.

In the following, $\mathcal{E}$ will often be a category such as *Set*.

Definition D.2. For $\mathcal{D}$ a non-associative category, its **graph hom-morphism** $\mathcal{D} : \mathcal{D}^{\mathrm{op}} \boxtimes \mathcal{D} \rightharpoonup Set$ is the graph morphism defined component-wise with:
$$\mathcal{D}(f, A)(g) = g \circledcirc f \qquad \mathcal{D}(A, f)(g) = f \circledcirc g$$

Definition D.3. Let $F : \mathcal{D} \rightharpoonup \mathcal{E}$ and $G : \mathcal{E} \rightharpoonup \mathcal{D}$ two graph morphisms between non-associative categories. An **adjunction between graph morphisms** (notation $F : \mathcal{D} \rightleftharpoons \mathcal{E} : G$) is given by a natural isomorphism of graph morphisms:
$$\varphi : \mathcal{E}(F-, =) \xrightarrow{\cong} \mathcal{D}(-, G=) : \mathcal{D}^{\mathrm{op}} \boxtimes \mathcal{E} \rightharpoonup Set$$
(i.e. natural component-wise). F is said to be left adjoint to G and G right adjoint to F.

The following proposition provides an alternative characterisation of an adjunction between graph morphisms:

Proposition D.4. *A adjunction between graph morphisms $F : \mathcal{D} \rightleftharpoons \mathcal{E} : G$ is the same thing as two pairs of natural transformations $\eta : \mathrm{Id}_{\mathcal{D}} \rightharpoonup GF$ and $\varepsilon : FG \rightharpoonup \mathrm{Id}_{\mathcal{E}}$ satisfying the following conditions:*

(1) inverses at a distance: *for all $f \in \mathcal{D}(A, GB)$ and $g \in \mathcal{E}(FA, B)$,*

$$G\varepsilon_B \circ (\eta_{GB} \circ f) = f$$
$$(g \circ \varepsilon_{FA}) \circ F\eta_A = g \ ,$$

(2) G-thunkability of η: *for all morphisms $f \in \mathcal{E}(FA, B), g \in \mathcal{E}(B, C)$ one has:*

$$G(g \circ f) \circ \eta_A = Gg \circ (Gf \circ \eta_A) \ ,$$

(3) F-linearity of ε: *for all morphisms $f \in \mathcal{D}(A, B), g \in \mathcal{D}(B, GC)$ one has:*

$$\varepsilon_C \circ F(g \circ f) = (\varepsilon_C \circ Fg) \circ Ff \ .$$

Proposition D.5. *Let an adjunction between graph morphisms $\varphi : \dot{\mathcal{E}}(F-, =) \xrightarrow{\cong} \dot{\mathcal{D}}(-, G=)$.*

(1) *F preserves thunkability, and G preserves linearity.*
(2) *For f, g morphisms in $\mathcal{D}$, one has $F(f \circ g) = Ff \circ Fg$ if and only if $\eta_X \circ f \circ g$ associates (for instance, whenever g is thunkable and whenever the codomain of f is positive).*
(3) *For f, g morphisms in $\mathcal{E}$, one has $G(f \circ g) = Gf \circ Gg$ if and only if $f \circ g \circ \varepsilon_X$ associates (for instance, whenever f is linear and whenever the domain of g is negative).*

Proposition D.6. *Let an adjunction between graph morphisms $\varphi : \dot{\mathcal{E}}(F-, =) \xrightarrow{\cong} \dot{\mathcal{D}}(-, G=)$.*

(1) *The following conditions are equivalent: F is functorial, φ preserves linearity, η is linear.*
(2) *The following conditions are equivalent: G is functorial, φ^{-1} preserves thunkability, ε is thunkable.*
(3) *If F is full and surjective on objects, then φ^{-1} preserves linearity (in particular ε is linear).*
(4) *If G is full and surjective on objects, then φ preserves thunkability (in particular η is thunkable).*

Proposition D.7. *Let $\mathcal{D}, \mathcal{E}$ two non-associative categories together with:*

- *for each $A \in \mathcal{D}$ an object $G_0 A \in \mathcal{E}$,*
- *a graph morphism $F : \mathcal{E} \rightharpoonup \mathcal{D}$ satisfying the following condition:*

$$\forall f \in \mathcal{E}(A, G_0 B), Ff \text{ is thunkable,} \tag{25}$$

- *for each $A \in \mathcal{E}$ and $B \in \mathcal{D}$ a family of bijections*

$$\varphi : \mathcal{D}(FA, B) \xrightarrow{\cong} \mathcal{E}(A, G_0 B)$$

  natural in A.

The graph morphism $G : \mathcal{D} \rightharpoonup \mathcal{E}$ defined with:

$$Gg \stackrel{\text{def}}{=} \varphi(g \circ \varphi^{-1}(\mathrm{id}_{G_0 B}))$$

is right adjoint to F.

Note that the condition (25) holds:

- whenever F preserves thunkability and $G_0 A$ is negative for every $A \in \mathcal{D}$,
- whenever F is negative.

Proof of Proposition D.7. The definition $Gg = \varphi(g \circ \varphi^{-1}(\mathrm{id}_{G_0 B}))$ defines G as a graph morphism $\mathcal{D} \rightharpoonup \mathcal{E}$ such that for each $A \in \mathcal{E}$ and $B \in \mathcal{D}$ there is a family of bijections

$$\varphi : \mathcal{D}(FA, B) \xrightarrow{\cong} \mathcal{E}(A, GB)$$

natural in A. Naturality in B then follows by hypothesis:

$$\begin{aligned}
Gg \circledcirc \varphi(f) &= \varphi(g \circledcirc \varphi^{-1}(\mathrm{id}_{G_0 B})) \circledcirc \varphi(f) && \text{by definition}\\
&= \varphi((g \circledcirc \varphi^{-1}(\mathrm{id}_{G_0 B})) \circledcirc F\varphi(f)) && \text{by naturality in } A \text{ of } \varphi\\
&= \varphi(g \circledcirc (\varphi^{-1}(\mathrm{id}_{G_0 B}) \circledcirc F\varphi(f))) && \text{since } F\varphi(f) \text{ is thunkable}\\
&= \varphi(g \circledcirc (\varphi^{-1}(\mathrm{id}_{G_0 B} \circledcirc \varphi(f)))) && \text{by naturality in } A \text{ of } \varphi\\
&= \varphi(g \circledcirc f)
\end{aligned}$$

□

Proposition D.8.

- *A positive shift on a non-associative category $\mathcal{D}$ is the same thing as a graph adjunction $\Downarrow : \mathcal{D} \rightleftharpoons \mathcal{D} : \mathrm{Id}_{\mathcal{D}}$ such that $\Downarrow A$ is positive every object A.*
- *A negative shift on a non-associative category $\mathcal{D}$ is the same thing as a graph adjunction $\mathrm{Id}_{\mathcal{D}} : \mathcal{D} \rightleftharpoons \mathcal{D} : \Uparrow$ such that $\Uparrow A$ is negative every object A.*

A duploid $\mathcal{D}$ is therefore the same thing as a non-associative category $\mathcal{D}$ where all objects are either positive or negative (or both), together with graph adjunctions:

$$\dot{\mathcal{D}}(\Downarrow -, =) \cong \dot{\mathcal{D}}(-, =) \cong \dot{\mathcal{D}}(-, \Uparrow =) : \mathcal{D}^{\mathrm{op}} \boxtimes \mathcal{D} \rightharpoonup \mathit{Set}$$

such that $\Downarrow A$ is positive and $\Uparrow A$ is negative for every object A.

Given a non-associative category $\mathcal{D}$, it is an instructive exercise to check that the data of a functor $\Downarrow : \mathcal{D}_t \to \mathcal{D}_l$ together with a natural isomorphism

$$\mathcal{D}_l(\Downarrow -, =) \cong \mathcal{D}(-, =) : \mathcal{D}_t^{\mathrm{op}} \times \mathcal{D}_l \to \mathit{Set}$$

does not suffice to define a positive shift on $\mathcal{D}$, despite being implied by it.

E Symmetric monoidal closed duploids

Definition E.1. A (positive) symmetric monoidal duploid is **closed** when the graph morphism $- \otimes A : \mathcal{D} \to \mathcal{D}$ has a right adjoint (written $A \multimap -$) for every object A, such that $A \multimap B$ is negative for every objects A and B.

As we have previously seen, this definition does not imply the functoriality for $\multimap$, nor is such a requirement expected. However, the adjunction requirement is quite strong as it implies the following functoriality properties:

Proposition E.2. *Let $\mathcal{D}$ a closed symmetric monoidal duploid. $\multimap$ extends into a graph morphism $\mathcal{D}^{\mathrm{op}} \boxtimes \mathcal{D} \rightharpoonup \mathcal{N}$ which is functorial on the following sub-categories: $\mathcal{P}^{\mathrm{op}} \boxtimes \mathcal{N}$ (i.e. $P \multimap - : \mathcal{N} \to \mathcal{N}$ and $- \multimap N : \mathcal{P}^{\mathrm{op}} \to \mathcal{N}$) and $\mathcal{D}_t^{\mathrm{op}} \times \mathcal{D}_l$ preserving linearity (i.e.: $\multimap : \mathcal{D}_t^{\mathrm{op}} \times \mathcal{D}_l \to \mathcal{N}_l$).*

Proof. Let $\mathcal{D}$ a closed symmetric monoidal duploid and let us note

$$\varphi_A : \dot{\mathcal{D}}(- \otimes A, =) \xrightarrow{\cong} \dot{\mathcal{D}}(-, A \multimap =) : \mathcal{D} \boxtimes \mathcal{D} \rightharpoonup \mathit{Set}$$

For all A, $A \multimap -$ preserves linearity by D.5 (1) and is functorial on both $\mathcal{D}_l$ and $\mathcal{N}$ (separately) by D.5 (3).

The action $f \multimap C$ on the left for a morphism $f \in \mathcal{D}(A, B)$ is defined with

$$\varphi_{A, B \multimap C, C}(\mathrm{ev}_{B,C} \circledcirc ((B \multimap C) \otimes f)) \in \mathcal{D}(B \multimap C, A \multimap C)$$

where $\mathrm{ev}_{B,C} = \varphi^{-1}_{B, B \multimap C, C}(\mathrm{id}_{B \multimap C})$. This defines a graph morphism $- \multimap C : \mathcal{D}^{\mathrm{op}} \rightharpoonup \mathcal{D}$ for every object C which is adjoint to itself on the right:

$$\dot{\mathcal{D}}(-, = \multimap C) \cong \dot{\mathcal{D}}^{\mathrm{op}}(- \multimap C, =) : \mathcal{D}^{\mathrm{op}} \boxtimes \mathcal{D}^{\mathrm{op}} \rightharpoonup \mathit{Set}$$

Thus by D.5 again it preserves linearity in $\mathcal{D}^{\mathrm{op}}$ (thunkability in $\mathcal{D}$) and is functorial on both $\mathcal{D}_t^{\mathrm{op}}$ and $\mathcal{P}^{\mathrm{op}}$ (separately).

Lastly, for $f \in \mathcal{D}(A, B)$ and $g \in \mathcal{D}(C, D)$ one has

$$(f \multimap D) \circledcirc (B \multimap g) = (A \multimap g) \circledcirc (f \multimap C)$$

whenever f is thunkable or g is linear. Therefore $\multimap$ gives rise to a bifunctor $(- \multimap =) : \mathcal{D}_t^{\mathrm{op}} \times \mathcal{D}_l \to \mathcal{D}_l$. □

Now recall that the call-by-value arrow is usually defined in call-by-push-value and in polarised logics with $\Downarrow(P \multimap \Uparrow Q)$ where P and Q are positive. We recover in this way the a closed structure on $\mathcal{P}$ in a usual sense.

Proposition E.3. *Every closed symmetric monoidal structure on a duploid $\mathcal{D}$ gives rise to a closed structure on the symmetric monoidal Freyd structure (in the sense of Power [2002]): a (categorical) right adjoint to the functor $\iota - \otimes P : \mathcal{P}_t \to \mathcal{P}$.*

Proof. Let $\mathcal{D}$ a closed symmetric monoidal duploid and define $P \multimap^{+} - \stackrel{\mathrm{def}}{=} \Downarrow(P \multimap I-) : \mathcal{P} \rightharpoonup \mathcal{P}_t$. We write I for inferrable inclusion functors. One has the following natural isomorphisms of graph morphisms:

$$\begin{aligned}
\mathcal{P}(\iota - \otimes P, =) &= \dot{\mathcal{D}}(I - \otimes P, I=) && \text{by definition}\\
&\cong \dot{\mathcal{D}}(I-, P \multimap \Uparrow=) && \text{by adjunction}\\
&= \mathcal{D}_t(I-, P \multimap \Uparrow=) && \text{by definition}\\
&\cong \mathcal{D}_t(I-, \Downarrow(P \multimap \Uparrow=)) && I \dashv \Downarrow : \mathcal{D}_t \to \mathcal{P}_t \text{ when restricted to thunkables}\\
&= \mathcal{P}_t(-, \Downarrow(P \multimap \Uparrow=)) && \text{by definition}
\end{aligned}$$

An adjunction between graph morphisms between categories is an adjunction in the usual sense and therefore we precisely have an adjunction $\mathcal{P}(\iota - \otimes P, =) \cong \mathcal{P}_t(-, P \multimap^{+} =)$. □

Definition E.4. A linear effect adjunction is given by a symmetric monoidal category $\mathcal{A}$ and a $\widehat{\mathcal{A}}$-category $\underline{\mathcal{B}}$ with powers of representable presheaves, together with an $\widehat{\mathcal{A}}$-adjunction $\underline{L} \dashv \underline{R} : \underline{\mathcal{A}} \to \underline{\mathcal{B}}$.

We also recall the following useful characterisation of linear effect adjunction given in Melliès [2012b].

Proposition E.5. *A linear effect adjunction is the same thing as a symmetric monoidal category $\mathcal{A}$ together with an adjunction $L \dashv R : \mathcal{B} \to \mathcal{A}$, a pseudo-action $\multimap : \mathcal{A}^{\mathrm{op}} \times \mathcal{B} \to \mathcal{B}$ of $\mathcal{A}^{\mathrm{op}}$ on $\mathcal{B}$, and a family of adjunctions*

$$L(- \otimes A) \dashv R(A \multimap -) : \mathcal{B} \to \mathcal{A}\,. \tag{26}$$

We recall the definition of a pseudo-action of a monoidal category $\mathcal{A}$ on a category $\mathcal{B}$ from Melliès [2012b].

Definition E.6. A pseudo-action of a monoidal category $(\mathcal{A}, \otimes, I)$ on a category $\mathcal{B}$ is a functor

$$* : \mathcal{A} \times \mathcal{B} \to \mathcal{B}$$

together with two natural isomorphisms

$$\delta^2_{A,A',B} : (A \otimes A') * B \to A' * (A * B) \qquad\qquad \delta^0_A : I * A \to A$$

subject to three coherence laws: of δ^2 with the associator for $\otimes$, and of δ^0 with each unitor of $\otimes$.

Theorem E.7. *For every linear effect adjunction, its associated symmetric monoidal duploid* $\mathbf{dupl}_{L,R}$ *is closed.*

Proof. Given a linear effect adjunction and $\mathcal{D} = \mathbf{dupl}_{L,R}$ its associated symmetric monoidal duploid, the adjunctions (26) correspond to a family of bijections between hom-sets

$$\mathcal{D}(A \otimes_{\mathcal{D}} B, C) \cong \mathcal{D}(A, B \multimap_{\mathcal{D}} C) \tag{27}$$

where we define

$$B \multimap_{\mathcal{D}} C \stackrel{\text{def}}{=} B^{+} \multimap C^{-}$$

and where we recall that $A \otimes_{\mathcal{D}} B$ is $A^{+} \otimes B^{+}$. This isomorphism is natural in $A \in \mathcal{D}$. Since $- \otimes_{\mathcal{D}} B$ preserves thunkability and $B \multimap_{\mathcal{D}} C$ is negative, by Proposition D.7 the family of objects $B \multimap_{\mathcal{D}} C$ extends into a graph morphism $(B \multimap_{\mathcal{D}} -) : \mathcal{D} \rightharpoonup \mathcal{D}$ and the family of bijections (27) into an adjunction $- \otimes_{\mathcal{D}} B : \mathcal{D} \rightleftharpoons \mathcal{D} : B \multimap_{\mathcal{D}} -$. □

In analogy with the L-calculus, Proposition D.7 extends the stack constructor $V{\cdot}S$ from the CBPV target into a context constructor $V{\cdot}e$ by ς-expansion (focusing): $V{\cdot}e \stackrel{\text{def}}{=} \tilde{\mu}x.\langle \mu\alpha.\langle x \parallel V{\cdot}\alpha\rangle \parallel e\rangle$.

Conversely, given $\mathcal{D}$ a symmetric monoidal duploid, we have seen that the family of adjunctions $- \otimes_{\mathcal{D}} B : \mathcal{D} \rightleftharpoons \mathcal{D} : B \multimap_{\mathcal{D}} -$ gives rise to a bifunctor $\multimap : \mathcal{D}_t^{\text{op}} \times \mathcal{D}_l \to \mathcal{D}_l$. In fact, one easily sees from this adjunction that $\multimap$ has the structure of a pseudo-action of $\mathcal{D}_t^{\text{op}}$ on $\mathcal{D}_l$. Moreover, one has $\Uparrow(- \otimes A) \dashv \Downarrow(A \multimap -) : \mathcal{D}_l \to \mathcal{D}_t$ since $\Uparrow$ (resp. $\Downarrow$) is equivalent to $\mathrm{Id}_{\mathcal{D}_l}$ on $\mathcal{D}_l$ (resp. $\mathcal{D}_t$). Thus:

Theorem E.8. *Every symmetric monoidal closed duploid gives rise to an* $\mathbf{IMLL}_p^{\eta}$ *model on the adjunction (14).*

We expect that the monoidal structure can be obtained similarly as a left adjoint; however this would require either a suitable notion of "multi"-duploids or of closed duploid. Then the structure of shifts on duploids can be derived with $\Downarrow A = A \otimes 1$ and $\Uparrow A = 1 \multimap A$ rather than assumed.

F Linearly distributive duploids

We want to describe the structure inherited by a duploid associated to an adjunction of the form (2) where both categories $\mathscr{A}$ and $\mathscr{B}$ come equipped with symmetric monoidal structures noted $(\mathscr{A}, \oslash, \mathit{true})$ and $(\mathscr{B}, \varovee, \mathit{false})$, generalising linearly-distributive categories [Cockett and Seely 1997], in the sense that there are four distributivity laws (or commutators)

$$\begin{aligned}
\mathit{ldistr}^{\oslash}_{A_1,A_2,B} &: A_1 \oslash R(L(A_2) \varovee B) \to R(L(A_1 \oslash A_2) \varovee B)\\
\mathit{ldistr}^{\varovee}_{A,B_1,B_2} &: L(R(B_1 \varovee B_2) \oslash A) \to B_1 \varovee L(R(B_2) \oslash A)\\
\mathit{rdistr}^{\oslash}_{A_1,A_2,B} &: R(B \varovee L(A_1)) \oslash A_2 \to R(B \varovee L(A_1 \oslash A_2))\\
\mathit{rdistr}^{\varovee}_{A,B_1,B_2} &: L(A \oslash R(B_1 \varovee B_2)) \to L(A \oslash R(B_1)) \varovee B_2
\end{aligned}$$

introduced in Melliès [2017a] and assumed to make a number of coherence diagrams commute. We note that the strengths for $\oslash$ and $\varovee$ can be deduced from the commutators.

When translating the commutators into the duploid framework, the four rules collapse into only two, as they were merely cases depending on the polarity of A'/B. $\mathit{ldistr}^{\oslash}$ and $\mathit{rdistr}^{\varovee}$ become δ^l and $\mathit{ldistr}^{\varovee}$ and $\mathit{rdistr}^{\oslash}$ become δ^r.

Definition F.1. A **linearly distributive duploid** $\mathcal{D}$ is a duploid equipped with a pair of positive and negative symmetric monoidal structures related by two families of mappings:

$$\begin{aligned}
\delta^l_{A,B,C} &: A \otimes (B \parr C) \to (A \otimes B) \parr C\\
\delta^r_{A,B,C} &: (A \parr B) \otimes C \to A \parr (B \otimes C)
\end{aligned}$$

natural for each component and that respects the usual coherence diagrams for a linearly distributive category.

Definition F.2. Let $\mathcal{D}$ be a linearly distributive duploid. We say that a morphism $h \in \mathcal{D}(A \otimes B, C)$ is **linear wrt. A** (and we note it $h \in \mathcal{D}(\underline{A} \otimes B, C)$) when, for all $g \in \mathcal{D}(A', A)$ and $f \in \mathcal{D}(A'', A')$, we have

$$h \bullet ((g \odot f) \ltimes B) = h \bullet (g \ltimes B) \bullet (f \ltimes B).$$

Dually, we say that a morphism f from A to $B \parr C$ is **thunkable wrt. B** (noted $f \in \mathcal{D}(A, \underline{B} \parr C)$) when, for all $g \in \mathcal{D}(B, B')$ and $h \in \mathcal{D}(B', B'')$, we have

$$((h \odot g) \ltimes C) \circ f = (h \ltimes C) \circ (g \ltimes C) \circ f.$$

Proposition F.3. *Let h be a morphism from $A \otimes B$ to C. If A is positive, then h is linear wrt. A. Symmetrically, if B is negative, then $f \in \mathcal{D}(A, B \parr C)$ is thunkable wrt. B.*

G Dialogue duploids and dialogue functors

We start by describing explicitly the commuting diagrams expressing the naturality and coherence conditions in the definition of dialogue duploid given in def. 8.3.

$$\begin{array}{ccc} A' & \xrightarrow{h_A} & A \\ & {\scriptstyle \chi_{A',B,C}(f\bullet(h_A\ltimes B))} \searrow \quad \swarrow {\scriptstyle \chi_{A,B,C}(f)} & \\ & B^* \parr C & \end{array} \qquad \begin{array}{ccc} & A & \\ {\scriptstyle \chi_{A,B,C}(f)} \swarrow & & \searrow {\scriptstyle \chi_{A,B',C}(f\bullet(A\rtimes h_B))} \\ B^* \parr C & \xrightarrow{h_B^*\ltimes C} & B'^* \parr C \end{array} \qquad \begin{array}{ccc} & A & \\ {\scriptstyle \chi_{A,B,C}(f)} \swarrow & & \searrow {\scriptstyle \chi_{A,B,C'}(h_C\odot f)} \\ B^* \parr C & \xrightarrow{B^*\rtimes h_C} & B^* \parr C' \end{array} \tag{28}$$

$$\begin{array}{ccccc} \mathcal{D}(A \otimes (B \otimes C), D) & & \xrightarrow{\qquad\chi_{A,B\otimes C,D}\qquad} & & \mathcal{D}(A, (B \otimes C)^* \parr D) \\ \text{associativity} \downarrow & & & & \downarrow \text{monoidality, symmetry, associativity} \\ \mathcal{D}((A \otimes B) \otimes C, D) & \xrightarrow{\chi_{A\otimes B,C,D}} & \mathcal{D}(A \otimes B, C^* \parr D) & \xrightarrow{\chi_{A,B,C^*\parr D}} & \mathcal{D}(A, B^* \parr (C^* \parr D)) \end{array} \tag{29}$$

We define the notion of dialogue duploid functors.

Definition G.1. A **dialogue duploid functor** $F : \mathcal{D} \to \mathcal{D}'$ is a duploid functor, lax monoidal for $\otimes$ and colax monoidal for $\parr$ (i.e. F^{op} is lax monoidal) and equipped with a family of natural central invertible morphisms $\widetilde{F}_A : F(A^*) \cong (FA)^\star$ such that the following coherence diagram commutes:

$$\begin{array}{ccc} \mathcal{D}(A \otimes B, C) & \xrightarrow{\qquad\chi_{A,B,C}\qquad} & \mathcal{D}(B, A^* \parr C) \\ F \downarrow & & \downarrow F \\ \mathcal{D}'(F(A \otimes B), FC) & & \mathcal{D}'(FB, F(A^* \parr C)) \\ & & \downarrow \text{monoidality } (\parr) \text{ of } F \\ \text{monoidality } (\otimes) \text{ of } F \downarrow & & \mathcal{D}'(FB, F(A^*) \parr' FC) \\ & & \downarrow \widetilde{F}_A \\ \mathcal{D}'(FA \otimes' FB, FC) & \xrightarrow{\chi'_{FA,FB,FC}} & \mathcal{D}'(FB, (FA)^\star \parr' FC) \end{array}$$

Definition G.2. Dia$\mathcal{D}$upl is the category whose objects are the dialogue duploids and whose morphisms are the dialogue duploid functors.

H Interpretation of the syntax

This section and the one that follows adapts to the classical case the technique used in [CFMM 2016] which is further detailed in Munch-Maccagnoni [2017]. A context on the left $(a_1 : A_1, a_2 : A_2, \ldots, a_n : A_n)$ is interpreted as $A_1 \otimes A_2 \otimes \cdots \otimes A_n$ and a context on the right $(\beta_1 : B_1, \beta_2 : B_2, \ldots, \beta_n : B_n)$ is interpreted as $B_1 ⅋ B_2 ⅋ \cdots ⅋ B_n$.

Let Γ and Γ' be two contexts on the left and σ an element of $\Sigma(\Gamma, \Gamma')$. We note $[\![\sigma]\!]$ the associated canonical isomorphism of $\mathcal{D}(\Gamma, \Gamma')$ constructed by composing symmmetries of $\otimes$. We stress on the fact that, as a composition of thunkable morphisms, $[\![\sigma]\!]$ is thunkable. Dually, let Δ and Δ' be two contexts on the right and $\tilde{\sigma}$ an element of $\Sigma(\Delta', \Delta)$. The associated canonical isomorphism of $\mathcal{D}(\Delta', \Delta)$ obtained by composing symmetries of ⅋ is noted $[\![\tilde{\sigma}]\!]$ and is linear.

H.1 Interpretation of judgements

- $[\![\Gamma \vdash t : A \mid \Delta]\!] \in \mathcal{D}(\Gamma, A ⅋ \Delta)$
- $[\![\Gamma \vdash V : A \mid \Delta]\!] \in \mathcal{D}(\Gamma, \underline{A} ⅋ \Delta)$
- $[\![\Gamma \mid e : A \vdash \Delta]\!] \in \mathcal{D}(\Gamma \otimes A, \Delta)$
- $[\![\Gamma \mid S : A \vdash \Delta]\!] \in \mathcal{D}(\Gamma \otimes \underline{A}, \Delta)$
- $[\![\mathrm{c} : (\Gamma \vdash \Delta)]\!] \in \mathcal{D}(\Gamma, \Delta)$

See def. F.2 for the meaning of the notation $\underline{A}$.

H.2 Interpretation of typing rules

Identity rules.

- $[\![a : A \vdash a : A \mid]\!] = \mathrm{id}_A \in \mathcal{D}_t(A, A)$
- $[\![\mid \alpha : A \vdash \alpha : A]\!] = \mathrm{id}_A \in \mathcal{D}_l(A, A)$
- $[\![\Gamma \mid \tilde{\mu} a^{\varepsilon}.\mathrm{c} : A_\varepsilon \vdash \Delta]\!] = [\![\mathrm{c} : (\Gamma, x : A_\varepsilon \vdash \Delta)]\!] \in \mathcal{D}(\Gamma \otimes A_\varepsilon, \Delta)$
- $[\![\Gamma \vdash \mu\alpha^{\varepsilon}.\mathrm{c} : A_\varepsilon \mid \Delta]\!] = [\![\mathrm{c} : (\Gamma \vdash \alpha : A_\varepsilon, \Delta)]\!] \in \mathcal{D}(\Gamma, A_\varepsilon ⅋ \Delta)$
- $[\![\langle t \parallel e\rangle^{\varepsilon} : (\Gamma, \Gamma' \vdash \Delta, \Delta')]\!]$
  $= ([\![\Gamma \mid e : A \vdash \Delta]\!] \ltimes \Delta') \circ (\delta^{l}_{\Gamma,A,\Delta'} \bullet (\Gamma \rtimes [\![\Gamma' \vdash t : A \mid \Delta']\!])) \in \mathcal{D}(\Gamma \otimes \Gamma', \Delta ⅋ \Delta')$
  where $\delta^{l}_{\Gamma,A,\Delta'} : \Gamma \otimes (A ⅋ \Delta') \to (\Gamma \otimes A) ⅋ \Delta'$ is the distributor.

Structural rules. $\forall \sigma \in \Sigma(\Gamma', \Gamma),\ \forall \tilde{\sigma} \in \Sigma(\Delta, \Delta')$

- $[\![\Gamma' \vdash t[\sigma, \tilde{\sigma}] : A \mid \Delta']\!] = (A \rtimes [\![\tilde{\sigma}]\!]) \circ ([\![\Gamma \vdash t : A \mid \Delta]\!] \circledcirc [\![\sigma]\!]) \in \mathcal{D}(\Gamma', A ⅋ \Delta')$
- $[\![\Gamma' \mid e[\sigma, \tilde{\sigma}] : A \vdash \Delta']\!] = [\![\tilde{\sigma}]\!] \circledcirc ([\![\Gamma \mid e : A \vdash \Delta]\!] \bullet ([\![\sigma]\!] \ltimes A)) \in \mathcal{D}(\Gamma' \otimes A, \Delta')$
- $[\![c[\sigma, \tilde{\sigma}] : (\Gamma' \vdash \Delta')]\!] = [\![\tilde{\sigma}]\!] \circledcirc ([\![\mathrm{c} : (\Gamma \vdash \Delta)]\!] \circledcirc [\![\sigma]\!]) \in \mathcal{D}(\Gamma', \Delta')$

Conjunction rules.

- $[\![\vdash () : 1 \mid]\!] = \mathrm{id}_1 \in \mathcal{D}_t(1, 1)$
- $[\![\Gamma \mid \tilde{\mu}().\mathrm{c} \vdash \Delta]\!] = [\![\mathrm{c} : (\Gamma \vdash \Delta)]\!] \circledcirc \rho_\Gamma \in \mathcal{D}(\Gamma \otimes \underline{1}, \Delta)$
  where $\rho_\Gamma : \Gamma \otimes 1 \to \Gamma$ is the right unitor of $\otimes$.
- $[\![\Gamma, \Gamma' \vdash V {\otimes} W : A \otimes B \mid \Delta, \Delta']\!]$
  $= ((((\sigma_{B,A} \ltimes \Delta) \circ \delta^{l}_{B,A,\Delta}) \bullet \sigma_{A⅋\Delta,B}) \ltimes \Delta') \circ \delta^{l}_{A⅋\Delta,B,\Delta'}) \bullet ([\![\Gamma \vdash V : A \mid \Delta]\!] \otimes [\![\Gamma' \vdash W : B \mid \Delta']\!])$
  $\in \mathcal{D}(\Gamma \otimes \Gamma', \underline{(A \otimes B)} ⅋ \Delta ⅋ \Delta')$
- $[\![\Gamma \mid \tilde{\mu}(a {\otimes} b).\mathrm{c} : A \otimes B \vdash \Delta]\!] = [\![\Gamma, a : A, b : B \vdash \Delta]\!] \in \mathcal{D}(\Gamma \otimes \underline{A \otimes B}, \Delta)$

Disjunction rules.

- $[\![\mid [] : \bot \vdash]\!] = \mathrm{id}_\bot \in \mathcal{D}_l(\bot, \bot)$
- $[\![\Gamma \vdash \mu[].\mathrm{c} \mid \Gamma]\!] = \lambda'_\Delta \circledcirc [\![\mathrm{c} : (\Gamma \vdash \Delta)]\!] \in \mathcal{D}(\Gamma, \underline{\bot} ⅋ \Delta)$
  where $\lambda'_\Delta : \Delta \to \bot ⅋ \Delta$ is the left unitor of ⅋.
- $[\![\Gamma, \Gamma' \mid S ⅋ S' : A ⅋ B \vdash \Delta, \Delta']\!]$

$= [\![\Gamma \mid S : A \vdash \Delta]\!] ⅋ [\![\Gamma' \mid S' : B \vdash \Delta']\!] \circ (\delta^l_{\Gamma, A, (\Gamma' \otimes B)} \bullet (\Gamma \otimes (\sigma'_{(\Gamma' \otimes B), A} \circ \delta^l_{\Gamma', B, A})) \bullet (\Gamma \otimes \Gamma' \otimes \sigma'_{A,B}))$
$\in \mathcal{D}(\Gamma \otimes \Gamma' \otimes \underline{(A ⅋ B)}, \Delta ⅋ \Delta')$

- $[\![\Gamma \vdash \mu(\alpha ⅋ \beta).\mathrm{c} : A ⅋ B \mid \Delta]\!] = [\![\mathrm{c} : (\Gamma \vdash \alpha : A, \beta : B, \Delta)]\!] \in \mathcal{D}(\Gamma, \underline{A ⅋ B} ⅋ \Delta)$

Negation rules.

- $[\![\Gamma \vdash [S] : N^* \mid \Delta]\!] = \chi_{\Gamma, N, \Delta}([\![\Gamma \mid S : N \vdash \Delta]\!]) \in \mathcal{D}(\Gamma, \underline{N^*} ⅋ \Delta)$
- $[\![\Gamma \mid [V] : P^* \vdash \Delta]\!] = \chi^{-1}_{\Gamma, P^*, \Delta}((\nu_P \ltimes \Delta) \circ [\![\Gamma \vdash V : P \mid \Delta]\!]) \in \mathcal{D}(\Gamma \otimes \underline{P^*}, \Delta)$
- $[\![\Gamma \mid \tilde{\mu}[\alpha].\mathrm{c} : N^* \vdash \Delta]\!] = \chi^{-1}_{\Gamma, N^*, \Delta}((\nu_N \ltimes \Delta) \circ [\![\mathrm{c} : (\Gamma \vdash \alpha : N, \Delta)]\!]) \in \mathcal{D}(\Gamma \otimes \underline{N^*}, \Delta)$
- $[\![\Gamma \vdash \mu[a].\mathrm{c} : P^* \mid \Delta]\!] = \chi_{\Gamma, P, \Delta}([\![\mathrm{c} : (\Gamma, a : P \vdash \Delta)]\!]) \in \mathcal{D}(\Gamma, \underline{P^*} ⅋ \Delta)$

where $\nu_A : A \to A^{**}$.

I Soundness of the interpretation

We follow again Munch-Maccagnoni [2017] which we adapt to classical logic with an involutive negation. We leave implicit the assignment of objects to atoms, and the interpretation of types as objects. We start by proving coherence properties of the interpretation. We say that two derivations are **equivalent** if their interpretation are equal in all dialogue duploids.

Lemma I.1. *For any typing derivations, there is an equivalent derivation starting by one structural rule.*

Proof. We treat the case of a typing derivation of $\Gamma \vdash t : A \mid \Delta$; the other cases are similar. We look at the smallest equivalent typing derivation of $\Gamma \vdash t : A \mid \Delta$ in terms of number of rules used. If it starts by two structural rules $\tau, \tilde{\tau}$ and $\sigma, \tilde{\sigma}$, then the derivation where the two first rules are replaced by the structural rule $\tau \circ \sigma, \tilde{\sigma} \circ \tilde{\tau}$ is equivalent and uses strictly less rules, which is impossible by hypothesis. So we have a derivation starting with at most one structural rule. If there is none, we can always add the structural rule of the identity, which is interpreted as the identity. □

For a term $\mathfrak{g}$, we will note $\mathbf{fv}\,\mathfrak{g}$ the set of free variables of $\mathfrak{g}$ and $\mathbf{fcv}\,\mathfrak{g}$ the set of free co-variables of $\mathfrak{g}$. For Γ a context and X a subset of the domain of Γ, we will note the restriction of Γ to X as $\Gamma_{\upharpoonright X}$.

Lemma I.2. *For any derivation* $\mathrm{c} : (\Gamma \vdash \Delta)$*, one has* $\mathbf{fv}\,\mathrm{c} = \mathrm{dom}\,\Gamma$ *and* $\mathbf{fcv}\,\mathrm{c} = \mathrm{dom}\,\Delta$ *and similarly for* t *and* e *replacing* c.

Proof. By induction on the derivation. □

We prove a *coherent generation lemma* which says that, from the form of the term, we can deduce the first rules of a derivation, or, at least, find an equivalent derivation starting by those rules.

Lemma I.3. $(\vdash \mathbf{ax})$ *: Any derivation of* $\Gamma \vdash x : A \mid \Delta$ *satisfies* $\Gamma = (x : A)$ *and* $\Delta = \emptyset$ *and is equivalent to the derivation:*

$$\frac{}{x : A \vdash x : A \mid}\ (\vdash \mathbf{ax})$$

($\mathbf{cut}^\varepsilon$) *: For any derivation of* $\langle t \parallel e\rangle^\varepsilon : (\Gamma \vdash \Delta)$*, there exists* A^ε *and an equivalent derivation ending with:*

$$\dfrac{\dfrac{\Gamma_{\restriction \mathbf{fv}\, e} \mid e : A^\varepsilon \vdash \Delta_{\restriction \mathbf{fcv}\, e} \qquad \Gamma_{\restriction \mathbf{fv}\, t} \vdash t : A^\varepsilon \mid \Delta_{\restriction \mathbf{fcv}\, t}}{\langle t \parallel e\rangle^\varepsilon : (\Gamma_{\restriction \mathbf{fv}\, e}, \Gamma_{\restriction \mathbf{fv}\, t} \vdash \Delta_{\restriction \mathbf{fcv}\, e}, \Delta_{\restriction \mathbf{fcv}\, t})}\ (\mathbf{cut}^\varepsilon)}{\langle t \parallel e\rangle^\varepsilon : (\Gamma \vdash \Delta)}\ (\sigma, \tilde\sigma)$$

where $\sigma \in \Sigma(\Gamma, (\Gamma_{\restriction \mathbf{fv}\, t}, \Gamma_{\restriction \mathbf{fv}\, e}))$ *is the unique permutation without renaming from* Γ *to* $(\Gamma_{\restriction \mathbf{fv}\, t}, \Gamma_{\restriction \mathbf{fv}\, e})$ *and* $\tilde\sigma \in \Sigma((\Delta_{\restriction \mathbf{fcv}\, t}, \Delta_{\restriction \mathbf{fcv}\, e}), \Delta)$ *is the unique permutation without renaming from* $(\Delta_{\restriction \mathbf{fcv}\, t}, \Delta_{\restriction \mathbf{fcv}\, e})$ *to* Δ*.*

($\vdash -^*$) *: For any derivation of* $\Gamma \vdash [S] : A \mid \Delta$*, one has* A *of the form* N^* *and an equivalent derivation ending with:*

$$\dfrac{\Gamma \mid S : N \vdash \Delta}{\Gamma \vdash [S] : N^* \mid \Delta}\ (\vdash -^*)$$

The other cases are similar.

Proof. ($\vdash$ ax) By using the previous lemma, we have that $\operatorname{dom}\Gamma = \{A\}$ and Δ is empty. We know from lem. I.1 that we can assume that it starts with one structural rule but it is a renaming which is interpreted as the identity. Finally, the only non-structural rule that can be applied to $x : A \vdash x : A \mid$ is ($\vdash$ ax).

($\vdash$ cut$^\varepsilon$) From the previous lemma, we know that $\operatorname{dom}\Gamma = \mathbf{fv}\langle t \parallel e\rangle^\varepsilon = \mathbf{fv}\, t \uplus \mathbf{fv}\, e$, so σ is well defined. We can say the same about Δ and $\tilde\sigma$. By using lem. I.1 and the fact that there is only one non-structural rule that can be applied to $\langle t \parallel e\rangle^\varepsilon$, we have a type A^ε and a derivation of $\langle t \parallel e\rangle^\varepsilon : (\Gamma \vdash \Delta)$ of the form:

$$\dfrac{\dfrac{\Gamma_1 \mid e[\tau, \tilde\tau] : A^\varepsilon \vdash \Delta_1 \qquad \Gamma_2 \vdash t[\tau, \tilde\tau] : A^\varepsilon \mid \Delta_2}{\langle t[\tau, \tilde\tau] \parallel e[\tau, \tilde\tau]\rangle^\varepsilon : (\Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2)}\ (\mathrm{cut}^\varepsilon)}{\langle t \parallel e\rangle^\varepsilon : (\Gamma \vdash \Delta)}\ (\tau, \tilde\tau)$$

We can add the structural rules $\sigma, \tilde\sigma$ and $\sigma^{-1}, \tilde\sigma^{-1}$ and, by centrality of symmetries and by coherence between symmetries and distributors, we can commute the cut rule and the structural rules to obtain the following equivalent derivation:

$$\dfrac{\dfrac{\dfrac{\Gamma_1 \mid e[\tau, \tilde\tau] : A^\varepsilon \vdash \Delta_1}{\Gamma_{\restriction \mathbf{fv}\, e} \mid e : A^\varepsilon \vdash \Delta_{\restriction \mathbf{fcv}\, e}}\ (\sigma^{-1} \circ \tau, \tilde\tau \circ \sigma^{-1}) \qquad \dfrac{\Gamma_2 \vdash t[\tau, \tilde\tau] : A^\varepsilon \mid \Delta_2}{\Gamma_{\restriction \mathbf{fv}\, t} \vdash t : A^\varepsilon \mid \Delta_{\restriction \mathbf{fcv}\, t}}\ (\sigma^{-1} \circ \tau, \tilde\tau \circ \sigma^{-1})}{\langle t \parallel e\rangle^\varepsilon : (\Gamma_{\restriction \mathbf{fv}\, e}, \Gamma_{\restriction \mathbf{fv}\, t} \vdash \Delta_{\restriction \mathbf{fcv}\, e}, \Delta_{\restriction \mathbf{fcv}\, t})}\ (\mathrm{cut}^\varepsilon)}{\langle t \parallel e\rangle^\varepsilon : (\Gamma \vdash \Delta)}\ (\sigma, \tilde\sigma)$$

($\vdash -^*$) : By using lem. I.1 and the fact that only the rule ($\vdash -^*$) can be applied, we have a negative type N and a derivation of the form:

$$\dfrac{\dfrac{\Gamma' \mid S[\tau, \tilde\tau] : N \vdash \Delta'}{\Gamma' \vdash [S[\tau, \tilde\tau]] : N^* \mid \Delta'}\ (\vdash -^*)}{\Gamma \vdash [S] : N^* \mid \Delta}\ (\tau, \tilde\tau)$$

We can commute the negation and the structural rule by naturality component-wise of χ and we obtain the equivalent derivation we seek:

$$\dfrac{\dfrac{\Gamma' \mid S[\tau, \tilde\tau] : N \vdash \Delta'}{\Gamma \mid S : N \vdash \Delta}\ (\tau, \tilde\tau)}{\Gamma \vdash [S] : N^* \mid \Delta}\ (\vdash -^*)$$

The other cases are similar and rely on the two previous lemmas and the coherence between the operations we are using. □

Thanks to the previous lemma, we can now reason on derivations up to equivalence by doing an induction on the structure of the term.

Lemma I.4. *We consider a derivation of* $\Gamma \vdash V : A \mid \Delta$ *and its interpretation* $\llbracket V \rrbracket \in \mathcal{D}(\Gamma, \underline{A} \parr \Delta)$.

- *For any derivation of* $\mathrm{c} : (\Gamma', a : A \vdash \Delta')$, *there exists a derivation of* $\mathrm{c}[V/a] : (\Gamma', \Gamma \vdash \Delta', \Delta)$ *such that:*
$$\llbracket c[V/a] \rrbracket = (\llbracket c \rrbracket \ltimes \Delta) \circ (\delta^l_{\Gamma', A, \Delta} \bullet (\Gamma' \rtimes \llbracket V \rrbracket))$$
- *For any derivation of* $\Gamma', a : A \vdash t : B \mid \Delta'$, *there exists a derivation of* $\Gamma', \Gamma \vdash t[V/a] : B \mid \Delta', \Delta$ *such that:*
$$\llbracket t[V/a] \rrbracket = (\llbracket t \rrbracket \ltimes \Delta) \circ (\delta^l_{\Gamma', A, \Delta} \bullet (\Gamma' \rtimes \llbracket t \rrbracket))$$
- *For any derivation of* $\Gamma', a : A \mid e : B \vdash \Delta'$, *there exists a derivation of* $\Gamma', \Gamma \mid e[V/a] : B \vdash \Delta', \Delta$ *such that:*
$$\llbracket e[V/x] \rrbracket = ((\llbracket e \rrbracket \bullet (\Gamma' \rtimes \sigma^{-1}_{A,B})) \ltimes \Delta) \circ (\delta^l_{\Gamma' \otimes B, A, \Delta} \bullet ((\Gamma' \otimes B) \rtimes \llbracket V \rrbracket) \bullet (\Gamma' \rtimes \sigma_{\Gamma, B}))$$

Proof. We reason by induction on c, t, e by using lem. I.3. In the case where the last rule used is $(\mathbf{cut}^\varepsilon)$ and c is of the form $\langle t \parallel e \rangle^\varepsilon$ with derivations $\Gamma'_{\restriction \mathbf{fv}\, t} \vdash t : B \mid \Delta'_{\restriction \mathbf{fcv}\, t}$ and $\Gamma'_{\restriction \mathbf{fv}\, e} \mid e : B \vdash \Delta'_{\restriction \mathbf{fcv}\, e}$: If $a \in \mathbf{fv}\, t$, by induction, we know that we have a derivation of $\Gamma'_{\restriction \mathbf{fv}\, t}, \Gamma \vdash t[V/a] : B \mid \Delta'_{\restriction \mathbf{fcv}\, t}, \Delta$ and that :
$$\llbracket t[V/a] \rrbracket = (\llbracket t \rrbracket \ltimes \Delta) \circ (\delta^l_{\Gamma'_{\restriction \mathbf{fv}\, t}, A, \Delta} \bullet (\Gamma'_{\restriction \mathbf{fv}\, t} \rtimes \llbracket V \rrbracket))$$

So,

$\llbracket \langle t \parallel e \rangle^\varepsilon [V/a] \rrbracket$

$= \llbracket \tilde{\sigma} \rrbracket \circ (((\llbracket e \rrbracket \ltimes (\Delta'_{\restriction \mathbf{fcv}\, t} \parr \Delta)) \circ (\delta^l_{\Gamma'_{\restriction \mathbf{fv}\, e}, B, \Delta'_{\restriction \mathbf{fcv}\, t} \parr \Delta} \bullet (\Gamma'_{\restriction \mathbf{fv}\, e} \rtimes \llbracket t[V/a] \rrbracket))) \bullet \llbracket \sigma \rrbracket)$

$= \llbracket \tilde{\sigma} \rrbracket \circ (((\llbracket e \rrbracket \ltimes (\Delta'_{\restriction \mathbf{fcv}\, t} \parr \Delta)) \circ (\delta^l_{\Gamma'_{\restriction \mathbf{fv}\, e}, B, \Delta'_{\restriction \mathbf{fcv}\, t} \parr \Delta} \bullet (\Gamma'_{\restriction \mathbf{fv}\, e} \rtimes ((\llbracket t \rrbracket \ltimes \Delta) \circ (\delta^l_{\Gamma'_{\restriction \mathbf{fv}\, t}, A, \Delta} \bullet (\Gamma'_{\restriction \mathbf{fv}\, t} \rtimes \llbracket V \rrbracket)))))) \bullet \llbracket \sigma \rrbracket)$

$= \llbracket \tilde{\sigma} \rrbracket \circ ((((\llbracket e \rrbracket \ltimes \Delta'_{\restriction \mathbf{fcv}\, t}) \circ (\delta^l_{\Gamma'_{\restriction \mathbf{fv}\, e}, B, \Delta'_{\restriction \mathbf{fcv}\, t}} \bullet (\Gamma'_{\restriction \mathbf{fv}\, e} \rtimes \llbracket t \rrbracket))) \ltimes \Delta) \circ (\delta^l_{(\Gamma'_{\restriction \mathbf{fv}\, e}, \Gamma'_{\restriction \mathbf{fv}\, t}), A, \Delta} \bullet ((\Gamma'_{\restriction \mathbf{fv}\, e}, \Gamma'_{\restriction \mathbf{fv}\, t}) \rtimes \llbracket V \rrbracket) \bullet \llbracket \sigma \rrbracket))$

by thunkability of V and δ^l

$= \llbracket \tilde{\sigma} \rrbracket \circ ((((\llbracket e \rrbracket \ltimes \Delta'_{\restriction \mathbf{fcv}\, t}) \circ (\delta^l_{\Gamma'_{\restriction \mathbf{fv}\, e}, B, \Delta'_{\restriction \mathbf{fcv}\, t}} \bullet (\Gamma'_{\restriction \mathbf{fv}\, e} \rtimes \llbracket t \rrbracket))) \ltimes \Delta) \circ ((\llbracket \sigma' \rrbracket \ltimes \Delta) \circ (\delta^l_{\Gamma', A, \Delta} \bullet (\Gamma' \rtimes \llbracket V \rrbracket))))$

by centrality and compatibility with the distributor of symmmetries

$= (\llbracket \tilde{\sigma}' \rrbracket \circ ((\llbracket e \rrbracket \ltimes \Delta'_{\restriction \mathbf{fcv}\, t}) \circ (\delta^l_{\Gamma'_{\restriction \mathbf{fv}\, e}, B, \Delta'_{\restriction \mathbf{fcv}\, t}} \bullet (\Gamma'_{\restriction \mathbf{fv}\, e} \rtimes \llbracket t \rrbracket))) \ltimes \Delta) \circ ((\llbracket \sigma' \rrbracket \ltimes \Delta) \circ (\delta^l_{\Gamma', A, \Delta} \bullet (\Gamma' \rtimes \llbracket V \rrbracket)))$

by linearity of $\llbracket \tilde{\sigma} \rrbracket$

$= (\llbracket \langle t \parallel e \rangle^\varepsilon \rrbracket \ltimes \Delta) \circ (\delta^l_{\Gamma', A, \Delta} \bullet (\Gamma' \rtimes \llbracket V \rrbracket))$

where:
$$\sigma \in \Sigma((\Gamma', \Gamma), (\Gamma'_{\restriction \mathbf{fv}\, e}, \Gamma'_{\restriction \mathbf{fv}\, t \setminus \{x\}}, \Gamma))$$
$$\sigma' \in \Sigma((\Gamma', x : A), (\Gamma'_{\restriction \mathbf{fv}\, e}, \Gamma'_{\restriction \mathbf{fv}\, t \setminus \{x\}}, x : A))$$
$$\tilde{\sigma} \in \Sigma((\Delta'_{\restriction \mathbf{fcv}\, e}, \Delta'_{\restriction \mathbf{fcv}\, t}, \Delta), (\Delta', \Delta))$$
$$\tilde{\sigma}' \in \Sigma((\Delta'_{\restriction \mathbf{fcv}\, e}, \Delta'_{\restriction \mathbf{fcv}\, t}), \Delta')$$

The other cases are also straightforward, by using induction and the compatibility of the operations we are using. □

The following lemma is exactly the symmetric of the previous and is proved accordingly.

Lemma I.5. *Let a derivation of* $\Gamma \mid S : A \vdash \Delta$. *We consider* $\llbracket S \rrbracket \in \mathcal{D}(\Gamma \otimes A, \Delta)$ *its interpretation.*

- *For any derivation of* $\mathrm{c} : (\Gamma' \vdash \alpha : A, \Delta')$*, there exists a derivation of* $\mathrm{c}[S/\alpha] : (\Gamma, \Gamma' \vdash \Delta, \Delta')$ *such that:*
$$[\![c[S/\alpha]]\!] = ([\![S]\!] \ltimes \Delta') \circ (\delta^l_{\Gamma,A,\Delta'} \bullet (\Gamma \rtimes [\![c]\!]))$$
- *For any derivation of* $\Gamma' \vdash t : B \mid \alpha : A, \Delta'$*, there exists a derivation of* $\Gamma, \Gamma' \vdash t[S/\alpha] : B \mid \Delta, \Delta'$ *such that:*
$$[\![t[S/\alpha]]\!] = ((\sigma'_{B,\Delta} \ltimes \Delta') \circ ([\![S]\!] \ltimes (B \parr \Delta')) \circ (\delta^l_{\Gamma,A,B\parr\Delta'} \bullet (\Gamma \rtimes ((\sigma'_{A,B} \ltimes \Delta')[\![t]\!]))$$
- *For any derivation of* $\Gamma' \mid e : B \vdash \alpha : A, \Delta'$*, there exists a derivation of* $\Gamma, \Gamma' \mid e[S/\alpha] : B \vdash \Delta, \Delta'$ *such that:*
$$[\![e[S/\alpha]]\!] = ([\![S]\!] \ltimes \Delta') \circ (\delta^l_{\Gamma,A,\Delta'} \bullet (\Gamma \rtimes [\![e]\!]))$$

We now prove the *sound subject reduction* lemma.

Lemma I.6. $\triangleright_{RE}$ *preserves typing, and, when restricted to typed terms,* $\triangleright_{RE}$ *preserves the interpretation.*

Proof. We reason by case analysis. We will treat in details the case of $(R-^*)$ and $(E-^*)$.

$(R-^*)$ For any $\mathrm{c} = \langle [S] \parallel \tilde{\mu}[\alpha].\mathrm{c}' \rangle^+ \triangleright_R \mathrm{c}'[S/\alpha]$ and derivation of $\mathrm{c} : (\Gamma \vdash \Delta)$, by applying lem. I.3, we have a negative type N and an equivalent derivation of the form:

$$\dfrac{\dfrac{\dfrac{\mathrm{c}' : (\Gamma_{\restriction \mathbf{fv}\, \mathrm{c}'} \vdash \alpha : N, \Delta_{\restriction \mathbf{fcv}\, \mathrm{c}' \setminus \{\alpha\}})}{\Gamma_{\restriction \mathbf{fv}\, \mathrm{c}'} \mid \tilde{\mu}[\alpha].\mathrm{c}' : N^* \vdash \Delta_{\restriction \mathbf{fcv}\, \mathrm{c}' \setminus \{\alpha\}}}\,(-^* \vdash) \quad \dfrac{\Gamma_{\restriction \mathbf{fv}\, S} \mid S : N \vdash \Delta_{\restriction \mathbf{fcv}\, S}}{\Gamma_{\restriction \mathbf{fv}\, S} \vdash [S] : N^* \mid \Delta_{\restriction \mathbf{fcv}\, S}}\,(\vdash -^*)}{\langle [S] \parallel \tilde{\mu}[\alpha].\mathrm{c}' \rangle^+ : (\Gamma_{\restriction \mathbf{fv}\, \mathrm{c}'}, \Gamma_{\restriction \mathbf{fv}\, S} \vdash \Delta_{\restriction \mathbf{fcv}\, \mathrm{c}' \setminus \{\alpha\}}, \Delta_{\restriction \mathbf{fcv}\, S})}\,(\mathrm{cut}^+)}{\langle [S] \parallel \tilde{\mu}[\alpha].\mathrm{c}' \rangle^+ : (\Gamma \vdash \Delta)}\,(\sigma, \tilde{\sigma})$$

where $\sigma \in \Sigma(\Gamma, (\Gamma_{\restriction \mathbf{fv}\, S}, \Gamma_{\restriction \mathbf{fv}\, \mathrm{c}}))$ is the unique permutation without renaming from Γ to $(\Gamma_{\restriction \mathbf{fv}\, S}, \Gamma_{\restriction \mathbf{fv}\, \mathrm{c}})$ and $\tilde{\sigma} \in \Sigma((\Delta_{\restriction \mathbf{fcv}\, S}, \Delta_{\restriction \mathbf{fcv}\, \mathrm{c} \setminus \{\alpha\}}), \Delta)$ is the unique permutation without renaming from $(\Delta_{\restriction \mathbf{fcv}\, S}, \Delta_{\restriction \mathbf{fcv}\, \mathrm{c} \setminus \{\alpha\}})$ to Δ.

So, by the previous lemma, we have a derivation of $\mathrm{c}'[S/\alpha] : (\Gamma \vdash \Delta)$. Moreover, one has:

$$[\![\langle [S] \parallel \tilde{\mu}[\alpha].\mathrm{c}' \rangle^+ : (\Gamma \vdash \Delta)]\!] = [\![\mathrm{c}'|S/\alpha] : (\Gamma \vdash \Delta)]\!]$$

The proof goes along the lines of the proof of lem. K.1.

$(E-^*)$ For any $\tilde{\mu}[\alpha].\langle [\alpha] \parallel S \rangle^+ \triangleright_R S$ and derivation of $\Gamma \mid \tilde{\mu}[\alpha].\langle [\alpha] \parallel S' \rangle^+ : N^* \vdash \Delta$, by applying lem. I.3, we have an equivalent derivation of the form:

$$\dfrac{\dfrac{\Gamma \mid S : N^* \vdash \Delta \quad \dfrac{\dfrac{}{\mid \alpha : N \vdash \alpha : N}\,(\mathrm{ax})}{\vdash [\alpha] : N^* \mid \alpha : N}\,(\vdash -^*)}{\langle [\alpha] \parallel S \rangle^+ : (\Gamma \vdash \alpha : N, \Delta)}\,(\mathrm{cut}^+)}{\Gamma \mid \tilde{\mu}[\alpha].\langle [\alpha] \parallel S \rangle^+ : N^* \vdash \Delta}\,(-^* \vdash)$$

So, one has:

$$\begin{aligned}
&[\![\Gamma \mid \tilde{\mu}[\alpha].\langle [\alpha] \parallel S \rangle^+ : N^* \vdash \Delta]\!] \\
&= \chi^{-1}_{\Gamma,N^*,\Delta}((\nu_N \ltimes \Delta) \circ [\![\langle [\alpha] \parallel S \rangle^+]\!]) \\
&= \chi^{-1}_{\Gamma,N^*,\Delta}((\nu_N \ltimes \Delta) \circ \sigma'_{\Delta,N} \circ ([\![S]\!] \parr N) \circ (\delta^l_{\Gamma,N^*,N} \bullet (\Gamma \rtimes [\![[\alpha]]\!]))) \\
&= \chi^{-1}_{\Gamma,N^*,\Delta}((N^{**} \rtimes [\![S]\!]) \circ (\nu_N \ltimes (\Gamma \otimes N^*)) \circ \sigma'_{\Gamma\otimes N^*,N} \circ (\delta^l_{\Gamma,N^*,N} \bullet (\Gamma \rtimes [\![[\alpha]]\!]))) \\
&= [\![S]\!] \bullet \chi^{-1}_{\Gamma,N^*,\Gamma\otimes N^*}((\nu_N \ltimes (\Gamma \otimes N^*)) \circ \sigma'_{\Gamma\otimes N^*,N} \circ (\delta^l_{\Gamma,N^*,N} \bullet (\Gamma \rtimes [\![[\alpha]]\!]))) && \text{by naturality of } \chi^{-1} \\
&= [\![\Gamma \mid S : N^* \vdash \Delta]\!]
\end{aligned}$$

The other cases are treated similarly, by using the coherent generation lemma and the sound value/stack substitution. □

THEOREM I.7. $\to_{RE}$ *preserves typing.*

PROOF. We reason by induction on $\to_{RE}$. On the base case, we use the previous lemma. On other cases, we use lem. I.3 and the induction hypothesis. □

J Syntactically thunkable and central expressions

In this section, we will prove the characterizations in the classical L-calculus of thunkable morphisms (lem. 10.3) and central morphisms (lem. 10.5).

J.1 Proof of the characterization of thunkable morphisms (lem. 10.3)

Let t be an expression. We will prove that the following properties are equivalent:

(1) For all c, $\langle t \parallel \tilde{\mu}x^{\varepsilon}.c\rangle^{\varepsilon} \simeq_{RE} c[t/x]$;
(2) For all c and e, $\langle t \parallel \tilde{\mu}x^{\varepsilon_1}.\langle \mu\alpha^{\varepsilon_2}.c \parallel e\rangle^{\varepsilon_2}\rangle^{\varepsilon_1} \simeq_{RE} \langle \mu\alpha^{\varepsilon_2}.\langle t \parallel \tilde{\mu}x^{\varepsilon_1}.c\rangle^{\varepsilon_1} \parallel e\rangle^{\varepsilon_2}$;
(3) For all c, e, q of polarity ε_1 and $\tilde{q}$ of polarity ε_2, $\langle t \parallel \tilde{\mu}q.\langle \mu\tilde{q}.c \parallel e\rangle^{\varepsilon_2}\rangle^{\varepsilon_1} \simeq_{RE} \langle \mu\tilde{q}.\langle t \parallel \tilde{\mu}q.c\rangle^{\varepsilon_1} \parallel e\rangle^{\varepsilon_2}$.

We added the property 3, as it is an intermediate step that simplifies the proof.

PROOF. ($1 \Rightarrow 2$) Let c be a command and e a context.

$$\begin{aligned}\langle t \parallel \tilde{\mu}x^{\varepsilon_1}.\langle \mu\alpha^{\varepsilon_2}.\underline{c} \parallel e\rangle^{\varepsilon_2}\rangle^{\varepsilon_1} &\simeq_{RE} \left\langle t \,\middle\|\, \tilde{\mu}x^{\varepsilon_1}.\langle \mu\alpha^{\varepsilon_2}.\langle x \parallel \tilde{\mu}x^{\varepsilon_1}.c\rangle^{\varepsilon_1} \parallel e\rangle^{\varepsilon_2}\right\rangle^{\varepsilon_1} && (R\tilde{\mu}^{\varepsilon_1})\\ &\simeq_{RE} \langle \mu\alpha^{\varepsilon_2}.\langle t \parallel \tilde{\mu}x^{\varepsilon_1}.c\rangle^{\varepsilon_1} \parallel e\rangle^{\varepsilon_2} && \text{by using 1.}\end{aligned}$$

($2 \Rightarrow 3$) Let c be a command, e a context, q of polarity ε_1 and $\tilde{q}$ of polarity ε_2.

$$\begin{aligned}&\langle t \parallel \tilde{\mu}q.\langle \underline{\mu\tilde{q}.c} \parallel e\rangle^{\varepsilon_2}\rangle^{\varepsilon_1}\\ &\simeq_{RE} \left\langle t \,\middle\|\, \tilde{\mu}q.\underline{\langle \mu\alpha^{\varepsilon_2}.\langle \mu\tilde{q}.c \parallel \alpha\rangle^{\varepsilon_2} \parallel e\rangle^{\varepsilon_2}}\right\rangle^{\varepsilon_1} && (E\mu^{\varepsilon_2})\\ &\simeq_{RE} \left\langle t \,\middle\|\, \underline{\tilde{\mu}q.\left\langle q \,\middle\|\, \tilde{\mu}x^{\varepsilon_1}.\left\langle \mu\alpha^{\varepsilon_2}.\langle x \parallel \tilde{\mu}q.\langle \mu\tilde{q}.c \parallel \alpha\rangle^{\varepsilon_2}\rangle^{\varepsilon_1} \,\middle\|\, e\right\rangle^{\varepsilon_2}\right\rangle^{\varepsilon_1}}\right\rangle^{\varepsilon_1} && (Rq) \text{ and } (R\tilde{\mu}^{\varepsilon_1})\\ &\simeq_{RE} \left\langle t \,\middle\|\, \tilde{\mu}x^{\varepsilon_1}.\left\langle \mu\alpha^{\varepsilon_2}.\langle x \parallel \tilde{\mu}q.\langle \mu\tilde{q}.c \parallel \alpha\rangle^{\varepsilon_2}\rangle^{\varepsilon_1} \,\middle\|\, e\right\rangle^{\varepsilon_2}\right\rangle^{\varepsilon_1} && (Eq)\\ &\simeq_{RE} \left\langle \mu\alpha^{\varepsilon_2}.\left\langle t \,\middle\|\, \underline{\tilde{\mu}x^{\varepsilon_1}.\langle x \parallel \tilde{\mu}q.\langle \mu\tilde{q}.c \parallel \alpha\rangle^{\varepsilon_2}\rangle^{\varepsilon_1}}\right\rangle^{\varepsilon_1} \,\middle\|\, e\right\rangle^{\varepsilon_2} && \text{by using 2.}\\ &\simeq_{RE} \left\langle \underline{\mu\alpha^{\varepsilon_2}.\langle t \parallel \tilde{\mu}q.\langle \mu\tilde{q}.c \parallel \alpha\rangle^{\varepsilon_2}\rangle^{\varepsilon_1}} \,\middle\|\, e\right\rangle^{\varepsilon_2} && (E\tilde{\mu}^{\varepsilon_1})\\ &\simeq_{RE} \left\langle \mu\tilde{q}.\left\langle \underline{\mu\alpha^{\varepsilon_2}}.\langle t \parallel \tilde{\mu}q.\langle \underline{\mu\tilde{q}}.c \parallel \alpha\rangle^{\varepsilon_2}\rangle^{\varepsilon_1} \,\middle\|\, \tilde{q}\right\rangle^{\varepsilon_2} \,\middle\|\, e\right\rangle^{\varepsilon_2} && (E\tilde{q})\\ &\simeq_{RE} \langle \mu\tilde{q}.\langle t \parallel \tilde{\mu}q.c\rangle^{\varepsilon_1} \parallel e\rangle^{\varepsilon_2} && (R\mu^{\varepsilon_2}) \text{ and } (R\tilde{q})\end{aligned}$$

($3 \Rightarrow 1$) We first need to prove two intermediary results. We introduce the following notations :

$$\{V\} := [] \mathbin{⅋} V \quad \mu\{\alpha\}.c := \mu(\beta \mathbin{⅋} \alpha).\langle \mu[].c \parallel \beta\rangle^{-}$$

If t verifies the property 3, then, for all context e, one has :

$$\left\langle t \,\middle\|\, \tilde{\mu}x^{\varepsilon}.\langle \mu\{\alpha\}.\langle x \parallel \alpha\rangle^{\varepsilon} \parallel e\rangle^{-}\right\rangle^{\varepsilon} \simeq_{RE} \langle \mu\{\alpha\}.\langle t \parallel \alpha\rangle^{\varepsilon} \parallel e\rangle^{-}$$

PROOF. One has

$$\left\langle t \,\middle\|\, \tilde{\mu}x^{\varepsilon}.\left\langle \mu(\beta \mathbin{⅋} \alpha).\langle \mu[].\langle x \parallel \alpha\rangle^{\varepsilon} \parallel \beta\rangle^{-} \,\middle\|\, e\right\rangle^{-}\right\rangle^{\varepsilon}$$

$$
\begin{array}{lr}
\simeq_{RE} \left\langle \mu(\beta \parr \alpha).\left\langle t \,\middle\|\, \tilde\mu x^{\varepsilon}.\langle \mu[].\langle x \parallel \alpha\rangle^{\varepsilon} \parallel \beta\rangle^{-}\right\rangle^{\varepsilon} \,\middle\|\, e\right\rangle^{-} & \text{by using 3.}\\
\simeq_{RE} \left\langle \mu(\beta \parr \alpha).\left\langle \mu[].\langle t \parallel \underline{\tilde\mu x^{\varepsilon}.\langle x \parallel \alpha\rangle^{\varepsilon}}\rangle^{\varepsilon} \,\middle\|\, \beta\right\rangle^{-} \,\middle\|\, e\right\rangle^{-} & \text{by using 3.}\\
\simeq_{RE} \left\langle \mu(\beta \parr \alpha).\langle \mu[].\langle t \parallel \alpha\rangle^{\varepsilon} \parallel \beta\rangle^{-} \,\middle\|\, e\right\rangle^{-} & (E\tilde\mu^{\varepsilon}) \quad \square
\end{array}
$$

For u a term, we have the following equation :

$$\mu\alpha^{\varepsilon}.\langle \mu\{\beta\}.\langle u \parallel \beta\rangle^{\varepsilon} \parallel \{\alpha\}\rangle^{-} \simeq_{RE} u$$

Proof. One has

$$
\begin{array}{lr}
\mu\alpha^{\varepsilon}.\left\langle \underline{\mu(\gamma \parr \beta)}.\langle \mu[].\langle u \parallel \beta\rangle^{\varepsilon} \parallel \gamma\rangle^{-} \,\middle\|\, [] \parr \alpha\right\rangle^{-} & \\
\simeq_{RE} \mu\alpha^{\varepsilon}.\langle \underline{\mu[]}.\langle u \parallel \alpha\rangle^{\varepsilon} \parallel []\rangle^{-} & (R\parr)\\
\simeq_{RE} \underline{\mu\alpha^{\varepsilon}.\langle u \parallel \alpha\rangle^{\varepsilon}} & (R\perp)\\
\simeq_{RE} u & (E\mu^{\varepsilon}) \quad \square
\end{array}
$$

Now, we can prove the result.

$$
\begin{array}{lr}
\langle t \parallel \tilde\mu x^{\varepsilon}.c\rangle^{\varepsilon} \simeq_{RE} \left\langle t \,\middle\|\, \tilde\mu x^{\varepsilon}.\underline{c[\mu\alpha^{\varepsilon}.\langle \mu\{\beta\}.\langle x \parallel \beta\rangle^{\varepsilon} \parallel \{\alpha\}\rangle^{-}/x]}\right\rangle^{\varepsilon} & \\
\simeq_{RE} \left\langle t \,\middle\|\, \tilde\mu x^{\varepsilon}.\langle \mu\{\beta\}.\langle x \parallel \beta\rangle^{\varepsilon} \parallel \tilde\mu y^{-}.c[\mu\alpha^{\varepsilon}.\langle y \parallel \{\alpha\}\rangle^{-}/x]\rangle\right\rangle^{\varepsilon} & (R\tilde\mu^{-})\\
\simeq_{RE} \langle \mu\{\beta\}.\langle t \parallel \beta\rangle^{\varepsilon} \parallel \underline{\tilde\mu y^{-}}.c[\mu\alpha^{\varepsilon}.\langle y \parallel \{\alpha\}^{-}\rangle/x]\rangle & \text{proved above}\\
\simeq_{RE} c[\mu\alpha^{\varepsilon}.\langle \mu\{\beta\}.\langle t \parallel \beta\rangle^{\varepsilon} \parallel \{\alpha\}\rangle^{-}/x] & (R\tilde\mu^{-})\\
\simeq_{RE} c[t/x] & \square
\end{array}
$$

J.2 Proof of the characterization of central morphisms (lem. 10.5)

Proof. One has

$$
\begin{array}{lr}
\langle t \parallel \tilde\mu q.\langle u \parallel \underline{\tilde\mu q'.c}\rangle^{\varepsilon_2}\rangle^{\varepsilon_1} & \\
\simeq_{RE} \left\langle t \,\middle\|\, \tilde\mu q.\underline{\langle u \parallel \tilde\mu x^{\varepsilon_2}.\langle x \parallel \tilde\mu q'.c\rangle^{\varepsilon_2}\rangle^{\varepsilon_2}}\right\rangle^{\varepsilon_1} & (E\tilde\mu^{\varepsilon_2})\\
\simeq_{RE} \left\langle t \,\middle\|\, \underline{\tilde\mu q.\left\langle q \,\middle\|\, \tilde\mu y^{\varepsilon_1}.\left\langle u \,\middle\|\, \tilde\mu x^{\varepsilon_2}.\langle y \parallel \tilde\mu q.\langle x \parallel \tilde\mu q'.c\rangle^{\varepsilon_2}\rangle^{\varepsilon_1}\right\rangle^{\varepsilon_2}\right\rangle^{\varepsilon_1}}\right\rangle^{\varepsilon_1} & (Rq) \text{ and } (R\tilde\mu^{\varepsilon_1})\\
\simeq_{RE} \left\langle t \,\middle\|\, \tilde\mu y^{\varepsilon_1}.\left\langle u \,\middle\|\, \tilde\mu x^{\varepsilon_2}.\langle y \parallel \tilde\mu q.\langle x \parallel \tilde\mu q'.c\rangle^{\varepsilon_2}\rangle^{\varepsilon_1}\right\rangle^{\varepsilon_2}\right\rangle^{\varepsilon_1} & (Eq)\\
\simeq_{RE} \left\langle u \,\middle\|\, \tilde\mu x^{\varepsilon_2}.\left\langle t \,\middle\|\, \underline{\tilde\mu y^{\varepsilon_1}.\langle y \parallel \tilde\mu q.\langle x \parallel \tilde\mu q'.c\rangle^{\varepsilon_2}\rangle^{\varepsilon_1}}\right\rangle^{\varepsilon_1}\right\rangle^{\varepsilon_2} & \text{by centrality of } t\\
\simeq_{RE} \left\langle u \,\middle\|\, \underline{\tilde\mu x^{\varepsilon_2}.\langle t \parallel \tilde\mu q.\langle x \parallel \tilde\mu q'.c\rangle^{\varepsilon_2}\rangle^{\varepsilon_1}}\right\rangle^{\varepsilon_2} & (E\tilde\mu_1^{\varepsilon})\\
\simeq_{RE} \left\langle u \,\middle\|\, \tilde\mu q'.\left\langle q' \,\middle\|\, \underline{\tilde\mu x^{\varepsilon_2}}.\langle t \parallel \underline{\tilde\mu q}.\langle x \parallel \tilde\mu q'.c\rangle^{\varepsilon_2}\rangle^{\varepsilon_1}\right\rangle^{\varepsilon_2}\right\rangle^{\varepsilon_2} & (Eq')\\
\simeq_{RE} \langle u \parallel \tilde\mu q'.\langle t \parallel \tilde\mu q.c\rangle^{\varepsilon_1}\rangle^{\varepsilon_2} & (R\tilde\mu^{\varepsilon_2}) \text{ and } (Rq') \quad \square
\end{array}
$$

K A direct equational proof of the Hasegawa-Thielecke theorem

Lemma K.1. *Let $f \in \mathcal{D}(B, C)$ and $g \in \mathcal{D}(A, B)$ be two morphisms of $\mathcal{D}$. Let us note*

$$\varphi_D = \chi^{-1}_{A,D^*,\perp} : \mathcal{D}(A, D^{**} \parr \perp) \xrightarrow{\cong} \mathcal{D}(A \otimes D^*, \perp)$$

We have:

$$f \odot g = \nu_C \odot (\lambda'_{C^{**}} \circ \varphi_C^{-1}(\varphi_B((\nu_B^{-1} \ltimes \bot) \circ (\lambda_B'^{-1} \odot g)) \bullet (A \rtimes f^*)))$$

Proof. One has:

$$\begin{aligned}
f \odot g &= f \odot ((\lambda'_B \circ \lambda_B'^{-1}) \odot g) \\
&= f \odot \lambda'_B \circ (\lambda_B'^{-1} \odot g) && \text{by linearity of } \lambda' \\
&= (\nu_C \odot (f^{**} \odot \nu_B^{-1})) \odot \lambda'_B \circ (\lambda_B'^{-1} \odot g) && \text{by naturality of } \nu \\
&= \nu_C \odot ((f^{**} \odot \nu_B^{-1}) \odot \lambda'_B \circ (\lambda_B'^{-1} \odot g)) && \text{by linearity of } \nu \\
&= \nu_C \odot (\lambda'_{C^{**}} \circ ((f^{**} \odot \nu_B^{-1}) \ltimes \bot) \circ (\lambda_B'^{-1} \odot g)) \\
&= \nu_C \odot (\lambda'_{C^{**}} \circ (f^{**} \ltimes \bot) \circ (\nu_B^{-1} \ltimes \bot) \circ (\lambda_B'^{-1} \odot g)) \\
&= \nu_C \odot (\lambda'_{C^{**}} \circ (f^{**} \ltimes \bot) \circ \varphi_B^{-1}(\varphi_B((\nu_B^{-1} \ltimes \bot) \circ (\lambda_B'^{-1} \odot g)))) \\
&= \nu_C \odot (\lambda'_{C^{**}} \circ \varphi_C^{-1}(\varphi_B((\nu_B^{-1} \ltimes \bot) \circ (\lambda_B'^{-1} \odot g)) \bullet (A \rtimes f^*))) && \text{by eq. (28)} \quad \square
\end{aligned}$$

Theorem K.2. *A morphism of $\mathcal{D}$ is thunkable if and only if it is central for $\otimes$.*

Proof. We know by definition that thunkable morphisms are central for $\otimes$, so we only have to prove that central morphisms are thunkable. Let $A, B, C, D \in |\mathcal{D}|$ and $f \in \mathcal{D}(C, D)$, $g \in \mathcal{D}(B, C)$ and $h \in \mathcal{D}(A, B)$ such that h is central for $\otimes$. Let us note as before

$$\varphi_E = \chi^{-1}_{B,E^*,\bot} : \mathcal{D}(B, E^{**} \parr \bot) \xrightarrow{\cong} \mathcal{D}(B \otimes E^*, \bot)$$

One has:

$$\begin{aligned}
&(f \odot g) \odot h \\
&= (\nu_D \odot (\lambda'_{D^{**}} \circ \varphi_D^{-1}(\varphi_C((\nu_C^{-1} \ltimes \bot) \circ (\lambda_C'^{-1} \odot g)) \bullet (B \rtimes f^*)))) \odot h && \text{by the previous lemma} \\
&= \nu_D \odot ((\lambda'_{D^{**}} \circ \varphi_D^{-1}(\varphi_C((\nu_C^{-1} \ltimes \bot) \circ (\lambda_C'^{-1} \odot g)) \bullet (B \rtimes f^*))) \odot h) && \text{by linearity of } \lambda' \\
&= \nu_D \odot (\lambda'_{D^{**}} \circ (\varphi_D^{-1}(\varphi_C((\nu_C^{-1} \ltimes \bot) \circ (\lambda_C'^{-1} \odot g)) \bullet (B \rtimes f^*)) \odot h)) && \text{by linearity of } \lambda' \\
&= \nu_D \odot (\lambda'_{D^{**}} \circ (\varphi_D^{-1}(\varphi_C((\nu_C^{-1} \ltimes \bot) \circ (\lambda_C'^{-1} \odot g)) \bullet (B \rtimes f^*) \bullet (h \ltimes D^*)))) && \text{by eq. (28)} \\
&= \nu_D \odot (\lambda'_{D^{**}} \circ (\varphi_D^{-1}(\varphi_C((\nu_C^{-1} \ltimes \bot) \circ (\lambda_C'^{-1} \odot g)) \bullet (h \ltimes D^*) \bullet (B \rtimes f^*)))) && \text{by centrality of } h \\
&= \nu_D \odot (\lambda'_{D^{**}} \circ (\varphi_D^{-1}(\varphi_C(((\nu_C^{-1} \ltimes \bot) \circ (\lambda_C'^{-1} \odot g)) \odot h) \bullet (B \rtimes f^*)))) && \text{by eq. (28)} \\
&= \nu_D \odot (\lambda'_{D^{**}} \circ (\varphi_D^{-1}(\varphi_C((\nu_C^{-1} \ltimes \bot) \circ ((\lambda_C'^{-1} \odot g) \odot h)) \bullet (B \rtimes f^*)))) && \text{by lin. of } \nu^{-1} \text{ preserved by } \ltimes \\
&= \nu_D \odot (\lambda'_{D^{**}} \circ (\varphi_D^{-1}(\varphi_C((\nu_C^{-1} \ltimes \bot) \circ (\lambda_C'^{-1} \odot (g \odot h))) \bullet (B \rtimes f^*)))) && \text{by linearity of } \lambda'^{-1} \\
&= f \odot (g \odot h) && \text{by the previous lemma}
\end{aligned}$$

So h is thunkable. This concludes the proof. □

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