Syntax and semantics of focalisation with relative monads and comonads

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Abstract

The logical principles of focalisation and polarisation can be used to design well-behaved term syntaxes for sequent calculus, which play a role as meta-languages for describing effectful computation. On the semantics side, this corresponds to an axiomatic and polarised notion of model of computation stated in terms of non-associative categories as well as adjunctions between "bare" functors (reflexive graph morphisms) over such non-associative categories.

In this paper, we study the special and delicate cases of resource and effect modalities in a general intuitionistic and linear setting: an exponential comonad ! (refining the necessity modality □) and a strong monad (written ◊). The starting point of our contribution is noticing that the completeness for a polarised syntax for ! and ◊ with respect to (co)monads in linear call-by-push-value models can be achieved if we move to *relative* (co)monads (Altenkirch, Chapman and Uustalu, 2015; Arkor and McDermott, 2024): more precisely, comonads relative to ↓ (the positive shift functor) for ! and monads relative to ↑ (the negative shift functor) for ◊.

These specialisations of the concept of relative (co)monad to call-by-push-value adjunctions recently appeared in Jiang, Xue and New (2025) and Melliès (2025, 2026). Yet the syntax we present arose from proof-theoretic consideration in Munch-Maccagnoni (2009) and Curien, Fiore and Munch-Maccagnoni (CFMM 2016), without the link with relative (co)monads being noticed at the time. Our first remark and explanation is thus that (co)monads relative to a call-by-push-value adjunction have been motivated previously from a proof-theoretic perspective in the context of focalisation, which also provides a meta-language for these concepts in an effectful setting.

We carry out the study of these modalities from the axiomatic, non-associative point of view. We recall the definition of adjunction between bare functors in this context, and establish correspondence results between this notion of adjunction and that of relative adjunction. This correspondence is then extended to linear-non-linear and strong versions of adjunctions as needed to model ! and ◊.

1. From focalisation and call-by-push-value to polarised calculi

In this paper, we are interested in the links between the semantics of effectful programs featuring linearity, and the proof theory of *intuitionistic multiplicative-additive linear sequent calculus*

$$
\begin{array}{rrcl}
\text{polarities} & \varepsilon & ::= & + \mid - \\
\text{types} & A, B, A_\varepsilon & ::= & P \mid N \\
\text{positive} & P, Q, A_+ & ::= & X^+ \mid \mathbf{1} \mid A \otimes B \mid A \oplus B \mid \mathbf{0} \\
\text{negative} & N, M, A_- & ::= & X^- \mid A \multimap B \mid A \,\&\, B \mid \top
\end{array}
$$

(a) Types

- Judgements are of the form $\Gamma \vdash A$
- Γ is an ordered list of formulæ $A_1, \ldots, A_n$.
- Γ, Γ' denotes the concatenation of Γ and Γ'.

(b) Judgements

$$
\frac{}{A \vdash A}\,(\mathrm{ax}) \qquad \frac{\Gamma \vdash A \quad \Gamma', A \vdash B}{\Gamma, \Gamma' \vdash B}\,(\mathrm{cut})
$$

(c) Identity

$$
\frac{\Gamma, A, B, \Gamma' \vdash C}{\Gamma, B, A, \Gamma' \vdash C}\,(\mathrm{ex})
$$

(d) Structure

$$
\frac{\Gamma \vdash A \quad \Gamma', B \vdash C}{\Gamma, \Gamma', A \multimap B \vdash C}\,(\multimap \vdash) \qquad \frac{\Gamma, A \vdash B}{\Gamma \vdash A \multimap B}\,(\vdash \multimap) \qquad \frac{}{\vdash \mathbf{1}}\,(\vdash \mathbf{1})
$$

$$
\frac{\Gamma \vdash A \quad \Gamma' \vdash B}{\Gamma, \Gamma' \vdash A \otimes B}\,(\vdash \otimes) \qquad \frac{\Gamma, A, B \vdash C}{\Gamma, A \otimes B \vdash C}\,(\otimes \vdash) \qquad \frac{\Gamma \vdash C}{\Gamma, \mathbf{1} \vdash C}\,(\mathbf{1} \vdash)
$$

$$
\frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \,\&\, B}\,(\vdash \&) \qquad \frac{\Gamma, A_{i \in \{1,2\}} \vdash C}{\Gamma, A_1 \,\&\, A_2 \vdash C}\,(\&_i \vdash_f) \qquad \frac{}{\Gamma \vdash \top}\,(\vdash \top)
$$

$$
\frac{\Gamma, A \vdash C \quad \Gamma, B \vdash C}{\Gamma, A \oplus B \vdash C}\,(\oplus \vdash) \qquad \frac{\Gamma \vdash A_{i \in \{1,2\}}}{\Gamma \vdash A_1 \oplus A_2}\,(\vdash \oplus_i) \qquad \frac{}{\Gamma, \mathbf{0} \vdash A}\,(\mathbf{0} \vdash)
$$

(e) Logic

Figure 1: **IMALL**

(**IMALL**, see Figure 1). On the syntactic side we consider a polarised version of **IMALL**, more precisely a linear and intuitionistic version of polarised classical logic described by Girard (1991) and Danos, Joinet and Schellinx (DJS 1997), which originated in a Curry-Howard interpretation of focusing proof search (Andreoli, 1992) with roots in linear logic (Girard, 1987). Systems arising from this perspective are polarised in the sense that a difference between positive and negative formulæ appears and is handled formally. This distinction between formulæ is similar to the one between types existing in the call-by-push-value model of programming effects (CBPV: Levy 1999, 2004).

On the semantic side, we consider a linear version of CBPV, the linear CBPV models (Fiore, 2011; Melliès, 2012b; CFMM 2016). Recall that linear CBPV models are enriched (or strong) adjunctions between a (symmetric monoidal and distributive) category of values and a category of co-values (cartesian closed relatively to $\mathscr{V}$ in a certain sense):

$$
\begin{tikzcd}
\mathscr{V} \arrow[r, bend left, "\uparrow"] \arrow[r, phantom, "\perp"] & \mathscr{S} \arrow[l, bend left, "\downarrow"]
\end{tikzcd} \tag{1}
$$

The interest in such situations arose after a long series of works starting from monads used to model side-effects (Moggi, 1989, 1991), which soon after shifted attention to adjoint situations decomposing monads as pictured above (Fiore, 1994; Power and Robinson, 1997; Thielecke, 1997; Levy, 1999, 2004, 2005), followed by various attempts to "linearise" it, including the aforementioned works and others such as Egger, Møgelberg and Simpson (2009). In fact the works on polarisation by Girard (1991, 1993) can themselves be considered historically as a source of inspiration for the adjoint point of view, via Benton and Wadler (1996) and Streicher and Reus (1998).

We avoid developing the full details of strength for the adjunction above. Being enriched (or strong) for the situation (1) means that additional structure is needed to express multiple formulæ on the left-hand side of sequents, obtained by asking the above adjunction to underlie a $\widehat{\mathscr{V}}$-enriched adjunction where $\widehat{\mathscr{V}}$ is the symmetric monoidal category of presheaves on $\mathscr{V}$ (CFMM 2016), or equivalently by asking that the adjoints are compatible with the monoidal product of $\mathscr{V}$ and with the action of $\mathscr{V}^{\mathrm{op}}$ on $\mathscr{S}$ in a manner defined in Melliès (2012b).

As is by now well understood, a similar situation as (1) underpins Girard's polarised logics (Girard, 1991, 1993), albeit in a manner which is both more specific—by pertaining to (constructive) classical, intuitionistic and linear logics: said differently to non-linear, linearly-used and fully linear continuations (respectively)—and more general, given that they integrate in the case of Girard (1993) several adjunctions in the picture above as opposed to a single one. Many people should be cited for establishing the links with CBPV, mainly via the relationships between polarised logics, negative translations, and continuation-passing style.[1]

But a strength of the proof-theoretical point of view symbolised by polarised logics is that they can be a source of inspiration for better understanding CBPV models. In this paper, we would like indeed to further understand such situations used in programming language semantics, particularly when comes the consideration of a comonad on $\mathscr{V}$ and/or a monad on $\mathscr{S}$ which are again strong in a suitable sense—these are important situations related to the study of resources and effects as modalities. Thus we find it useful to start by recalling the links between effectful semantics and polarised logics.

1.1. The deductive system associated to an adjunction

Proof theory is concerned with the formal study of the structure of deductive systems. Categorical proof theory has historically approached deductive systems as categories, given by:

1. a collection of objects (formulæ),
2. for each pair of objects A, B a set of morphisms from A to B (of proofs $A \vdash B$),
3. a well-typed and associative composition operation ($A \vdash B$ and $B \vdash C$ give $A \vdash C$), and
4. for each object A an identity morphism from A to A ($A \vdash A$), neutral for composition.

A useful perspective brought by proof theory is that an adjoint situation such as (1) can indeed be analysed as a deductive system, however with a composition which is not associative a priori. Given any (associative) adjoint situation

$$L : \mathscr{A} \rightleftarrows \mathscr{B} : R$$

(where this notation denotes that $R : \mathscr{B} \to \mathscr{A}$ is right adjoint to $L : \mathscr{A} \to \mathscr{B}$), one can define a deductive system $\mathbf{dupl}_{L,R}$ as follows:

$$|\mathbf{dupl}_{L,R}| = |\mathscr{A}| \uplus |\mathscr{B}| \qquad\qquad \mathbf{dupl}_{L,R}(A, B) = \mathscr{O}(A^{+}, B^{-})$$

[1] for instance Murthy, 1992; Lafont, Reus and Streicher, 1993; Thielecke, 1997; Streicher and Reus, 1998; Ogata, 2000; Curien and Herbelin, 2000; Selinger, 2001; Ogata, 2002; Hofmann and Streicher, 2002; Laurent, 2002, 2003; Laurent, Quatrini and Tortora de Falco, 2005; Melliès and Tabareau, 2010.

where

$$\mathscr{O} : \mathscr{A}^{\mathrm{op}} \times \mathscr{B} \to \mathit{Set}$$

is a chosen distributor of oblique morphisms of the adjunction, and where by definition

$$X^+ \stackrel{\text{def}}{=} \begin{cases} X & \text{if } X \in |\mathscr{A}| \\ RX & \text{if } X \in |\mathscr{B}| \end{cases} \qquad\qquad X^- \stackrel{\text{def}}{=} \begin{cases} LX & \text{if } X \in |\mathscr{A}| \\ X & \text{if } X \in |\mathscr{B}| \end{cases}$$

Objects in $\mathscr{A}$ are called *positive* in $\mathbf{dupl}_{L,R}$ and those in $\mathscr{B}$ *negative*. It is an interesting exercise to check that for all objects A, B, C one can define a composition

$$\mathbf{dupl}_{L,R}(C, B) \times \mathbf{dupl}_{L,R}(A, C) \to \mathbf{dupl}_{L,R}(A, B)$$

and that this composition is associative if and only if the adjunction $L : \mathscr{A} \rightleftarrows \mathscr{B} : R$ is idempotent, in the sense of its multiplication or co-multiplication being iso (Munch-Maccagnoni, 2014).

Non-associative categories, such as $\mathbf{dupl}_{L,R}$, are simply categories without the assumption that composition is associative. We use the notation $\circledcirc$ for composition operations that are not associative a priori.

Additional structure, such as the symmetric monoidal and distributive structure on $\mathscr{V}$ and the cartesian closed structure on $\mathscr{S}$ relative to $\mathscr{V}$ (1), allows us to interpret logical connectives. In fact:

Proposition 1 (CFMM 2016)**.** *The deductive system* $\mathbf{dupl}_{\uparrow,\downarrow}$ *associated to a linear CBPV adjunction* (1) *satisfies the rules of* **IMALL** *(Figure 1).*

From the point of view of deductive systems as non-associative categories, just as in categorical proof theory in general, the interpretation of logics into the models determine a notion of equality between derivations which we seek to understand and respect (e.g. during cut elimination). This is the point of view of polarised logics (Girard, 1991), which can still be described and analysed using the tools of category theory, motivating a theory of non-associative categories developed in Munch-Maccagnoni (2014) and Mangel, Melliès and Munch-Maccagnoni (MMMM 2026).

1.2. The calculus $\mathbf{IMALL}_p$: A polarised syntax for linear CBPV

The calculus $\mathbf{IMALL}_p$ (*polarised* **IMALL**) is defined in Figure 2 and 3. $\mathbf{IMALL}_p$ is defined à la Barendregt, starting with an untyped syntax of pseudo-terms à la Curry (Figure 2) out of which (proper) terms are those that possess a valid typing derivation in the sense of Figure 3. The equational theory $\simeq_{\mathrm{RE}}$ is obtained as the well-typed restriction of the compatible equivalence closure of reductions $\rhd_{\mathrm{R}}$ (Figure 2b, β-like) and expansions $\rhd_{\mathrm{E}}$ (Figure 2c, η-like). The calculus $\mathbf{IMALL}_p$ enjoys the following good properties (for more details see CFMM 2016; Munch-Maccagnoni, 2017).

As a type system, it expresses the logic **IMALL** (which is evident from Figure 3 and 4), and can be interpreted in any linear CBPV model along the lines of the proof-theoretic interpretation of Proposition 1. The interpretation into linear CBPV models is sound: any two terms related by $\simeq_{\mathrm{RE}}$ have the same interpretation in all models.

On the proof-theoretic side, the compatible closure $\to_{\mathrm{R}}$ of reduction $\rhd_{\mathrm{R}}$ is confluent. This is due to the fact that the reduction relation $\rhd_{\mathrm{R}}$ defines an orthogonal higher-order rewriting system (left-linear, no critical pairs). In fact, $\to_{\mathrm{R}}$ defines a noetherian (confluent, normalising) and semantics-preserving cut-elimination strategy for **IMALL**. Furthermore integrating expansions into the cut-elimination results, the $\to_{\mathrm{R}}$-normal and $\to_{\mathrm{E}}$-long terms are in correspondence with focused normal forms, from which follows the completeness of focusing proof search. In this sense, $\mathbf{IMALL}_p$ realises a Curry-Howard interpretation of focusing in correspondence with linear CBPV.

$$
\begin{array}{lcccccc}
 & \mathbf{1} & \otimes & \oplus\,(i\in\{1,2\}) & \multimap & \&\,(i\in\{1,2\}) & \top/\mathbf{0}
\end{array}
$$

Values:

$$V,W ::= x \mid \mu\alpha^{-}.c \mid \;() \;\mid\; V\otimes W \;\mid\; \iota_i(V) \;\mid \mu(x\cdot\alpha).c \mid \mu{<}\alpha.c;\beta.c'{>} \mid \mu{<}V{>}$$

Stacks:

$$S ::= \alpha \mid \bar{\mu}x^{+}.c \mid \bar{\mu}().c \mid \bar{\mu}(x\otimes y).c \mid \bar{\mu}[x.c \mid y.c'] \mid\; V\cdot S \;\mid\; \pi_i\cdot S \;\mid \bar{\mu}[S]$$

Expressions: $t,u ::= V \mid \mu\alpha^{+}.c$ Contexts: $e ::= S \mid \bar{\mu}x^{-}.c$ Commands: $c ::= \langle V \,\|\, e\rangle^{-} \mid \langle t \,\|\, S\rangle^{+}$

(a) Terms

$$
\begin{array}{rlll}
(R\bar{\mu}^{\varepsilon}): & \langle V \,\|\, \bar{\mu}x^{\varepsilon}.c\rangle^{\varepsilon} & \rhd_{\mathrm{R}} & c[V/x] \\
(R\mu^{\varepsilon}): & \langle \mu\alpha^{\varepsilon}.c \,\|\, S\rangle^{\varepsilon} & \rhd_{\mathrm{R}} & c[S/\alpha] \\
(R\mathbf{1}): & \langle () \,\|\, \bar{\mu}().c\rangle^{+} & \rhd_{\mathrm{R}} & c \\
(R\otimes): & \langle V\otimes W \,\|\, \bar{\mu}(x\otimes y).c\rangle^{+} & \rhd_{\mathrm{R}} & c[V/x, W/y] \\
(R\oplus): & \langle \iota_i(V) \,\|\, \bar{\mu}[x_1.c_1 \mid x_2.c_2]\rangle^{+} & \rhd_{\mathrm{R}} & c_i[V/x_i] \\
(R\multimap): & \langle \mu(x\cdot\alpha).c \,\|\, V\cdot S\rangle^{-} & \rhd_{\mathrm{R}} & c[V/x, S/\alpha] \\
(R\&): & \langle \mu{<}\alpha_1.c_1;\alpha_2.c_2{>} \,\|\, \pi_i\cdot S\rangle^{-} & \rhd_{\mathrm{R}} & c_i[S/\alpha_i]
\end{array}
$$

(no rules $R\top$, $R\mathbf{0}$)

(b) Reduction rules

$$
\begin{array}{rl}
(E\bar{\mu}^{\varepsilon}): & e \rhd_{\mathrm{E}} \bar{\mu}x^{\varepsilon}.\langle x \,\|\, e\rangle^{\varepsilon} \\
(E\mu^{\varepsilon}): & t \rhd_{\mathrm{E}} \mu\alpha^{\varepsilon}.\langle t \,\|\, \alpha\rangle^{\varepsilon} \\
(E\mathbf{1}): & S \rhd_{\mathrm{E}} \bar{\mu}().\langle () \,\|\, S\rangle^{+} \\
(E\otimes): & S \rhd_{\mathrm{E}} \bar{\mu}(x\otimes y).\langle x\otimes y \,\|\, S\rangle^{+} \\
(E\oplus): & S \rhd_{\mathrm{E}} \bar{\mu}[x.\langle \iota_1(x) \,\|\, S\rangle^{+} \mid y.\langle \iota_2(y) \,\|\, S\rangle^{+}] \\
(E\multimap): & V \rhd_{\mathrm{E}} \mu(x\cdot\alpha).\langle V \,\|\, x\cdot\alpha\rangle^{-} \\
(E\&): & V \rhd_{\mathrm{E}} \mu{<}\alpha.\langle V \,\|\, \pi_1\cdot\alpha\rangle^{-};\beta.\langle V \,\|\, \pi_2\cdot\beta\rangle^{-}{>} \\
(E\top): & V \rhd_{\mathrm{E}} \mu{<}x_1\otimes\cdots\otimes x_n{>} \\
(E\mathbf{0}): & S \rhd_{\mathrm{E}} \bar{\mu}[x_1\cdots x_n\cdot\alpha]
\end{array}
$$

(c) Extensionality rules

Figure 2: $\mathbf{IMALL}_p$: calculus

The syntax in fact also defines a syntactic model of linear CBPV, via a construction for which we need the following definition first.

Definition 2. A morphism f in a non-associative category is **thunkable** if any composition chain $h \odot g \odot f$ associates. A morphism h in a non-associative category is **linear** if any composition chain $h \odot g \odot f$ associates.

Thunkable and linear morphisms determine (associative) sub-categories of the non-associative category. We call $\mathscr{P}_t$ the subcategory of positive objects and thunkable maps of $\mathbf{IMALL}_p$ seen as a deductive system with $\simeq_{\mathrm{RE}}$ as equality of proofs, and $\mathscr{N}_l$ its subcategory of negative objects and linear maps. An advantage of considering thunkable and linear maps is that we do not need to define so-called "complex" values and stacks as in Levy (2004).

Proposition 3. *The* $\mathbf{IMALL}_p$ *calculus defines a syntactic linear CBPV model*

$$
\begin{tikzcd}
\mathscr{P}_t \arrow[rr, bend left, "\mathbf{1}\multimap-"] \arrow[rr, phantom, "\perp"] & & \mathscr{N}_l \arrow[ll, bend left, "\mathbf{1}\otimes-"]
\end{tikzcd}
$$

with the symmetric monoidal and distributive structure on $\mathscr{P}_t$ *given by the connectives* $\otimes, \mathbf{1}$ *and* $\oplus, \mathbf{0}$*, and the cartesian closure relative to* $\mathscr{P}_t$ *on* $\mathscr{N}_l$ *given by the connectives* $\&, \top$ *and* $\multimap$*.*

- Judgements are:
  $\Gamma \vdash t : A \mid \quad \Gamma \mid e : A \vdash \Delta \quad c : (\Gamma \vdash \Delta)$
- $\mathbf{\Gamma}$ is a map from a finite set of variables to types provided with a total order $\leq_\Gamma$ on its domain, notation $\Gamma = (x_1{:}A_1, \ldots, x_n{:}A_n)$.
- $\mathbf{\Delta}$ is a pair $\alpha : A$ of a co-variable and a type.
- Concatenation (Γ, Γ') is defined when the domains of Γ and Γ' are disjoint.

(a) Judgements

$$\frac{}{x : A \vdash x : A \mid}\,(\vdash \mathrm{ax}) \qquad \frac{}{\mid \alpha : A \vdash \alpha : A}\,(\mathrm{ax} \vdash)$$

$$\frac{c : (\Gamma, x : A_\varepsilon \vdash \Delta)}{\Gamma \mid \tilde{\mu} x^\varepsilon.c : A_\varepsilon \vdash \Delta}\,(\tilde{\mu}^\varepsilon \vdash) \qquad \frac{c : (\Gamma \vdash \alpha : A_\varepsilon)}{\Gamma \vdash \mu\alpha^\varepsilon.c : A_\varepsilon \mid}\,(\vdash \mu^\varepsilon)$$

$$\frac{\Gamma \vdash t : A_\varepsilon \mid \quad \Gamma' \mid e : A_\varepsilon \vdash \Delta}{\langle t \,\|\, e \rangle^\varepsilon : (\Gamma, \Gamma' \vdash \Delta)}\,(\mathrm{cut}^\varepsilon)$$

(b) Identity

- $\mathbf{\Sigma(\Gamma;\Gamma')}$ is the set of maps $\sigma : \mathrm{dom}\,\Gamma \to \mathrm{dom}\,\Gamma'$ satisfying $\Gamma'(\sigma(x)) = \Gamma(x)$ for all $x \in \mathrm{dom}\,\Gamma$. $\mathbf{\Sigma^*(\Gamma;\Gamma')}$ is the subset of $\Sigma(\Gamma;\Gamma')$ of maps that are bijective.
- The structural rules for $\mathbf{IMALL}_p$ are obtained by taking $\sigma \in \Sigma^*(\Gamma;\Gamma')$ below (renaming, exchange). By taking $\Sigma(\Gamma;\Gamma')$ for the range of structural maps (adding weakening and contraction) we obtain $\mathbf{LJ}_p$, in correspondence with call-by-push-value models (CFMM 2016).

$$\frac{\Gamma \vdash t : A \mid}{\Gamma' \vdash t[\sigma] : A \mid}\,(\vdash \sigma) \qquad \frac{\Gamma \mid e : A \vdash \Delta}{\Gamma' \mid e[\sigma] : A \vdash \Delta}\,(\sigma \vdash) \qquad \frac{c : (\Gamma \vdash \Delta)}{c[\sigma] : (\Gamma' \vdash \Delta)}\,(\sigma)$$

(c) Structure

$$\frac{\Gamma \vdash V : A \mid \quad \Gamma' \mid S : B \vdash \Delta}{\Gamma, \Gamma' \mid V{\cdot}S : A \multimap B \vdash \Delta}\,(\multimap \vdash_f) \qquad \frac{c : (\Gamma, x : A \vdash \alpha : B)}{\Gamma \vdash \mu(x{\cdot}\alpha).c : A \multimap B \mid}\,(\vdash \multimap)$$

$$\frac{\Gamma \vdash V : A \mid \quad \Gamma' \vdash W : B \mid}{\Gamma, \Gamma' \vdash V{\otimes}W : A \otimes B \mid}\,(\vdash_f \otimes) \qquad \frac{c : (\Gamma, x : A, y : B \vdash \Delta)}{\Gamma \mid \tilde{\mu}(x{\otimes}y).c : A \otimes B \vdash \Delta}\,(\otimes \vdash)$$

$$\frac{c : (\Gamma \vdash \alpha : A) \quad c' : (\Gamma \vdash \beta : B)}{\Gamma \vdash \mu{<}\alpha.c; \beta.c'{>} : A \,\&\, B \mid}\,(\vdash \&) \qquad \frac{\Gamma \mid S : A_i \vdash \Delta}{\Gamma \mid \pi_i{\cdot}S : A_1 \,\&\, A_2 \vdash \Delta}\,(\&_i \vdash_f)$$

$$\frac{c : (\Gamma, x : A \vdash \Delta) \quad c' : (\Gamma, y : B \vdash \Delta)}{\Gamma \mid \tilde{\mu}[x.c \mid y.c'] : A \oplus B \vdash \Delta}\,(\oplus \vdash) \qquad \frac{\Gamma \vdash V : A_i \mid}{\Gamma \vdash \iota_i(V) : A_1 \oplus A_2 \mid}\,(\vdash_f \oplus_i)$$

$$\frac{}{\vdash () : \mathbf{1} \mid}\,(\vdash \mathbf{1}) \qquad \frac{c : (\Gamma \vdash \Delta)}{\Gamma \mid \tilde{\mu}().c : \mathbf{1} \vdash \Delta}\,(\mathbf{1} \vdash) \qquad \frac{\Gamma \vdash V : A \mid}{\Gamma \vdash \mu{<}V{>} : \top \mid}\,(\vdash_f \top) \qquad \frac{\Gamma \mid S : A \vdash \Delta}{\Gamma \mid \tilde{\mu}[S] : \mathbf{0} \vdash \Delta}\,(\mathbf{0} \vdash_f)$$

(d) Logic

Figure 3: $\mathbf{IMALL}_p$: type system

We therefore have two way to organise $\mathbf{IMALL}_p$ categorically with cut as composition and $\simeq_{\mathrm{RE}}$ as equality of morphisms: 1) as deductive systems, i.e. non-associative categories, and 2) as linear CBPV model. The two interpretations turn out to be equivalent, in the sense that the non-associative category arising from the adjunctions between thunkables and linears is equivalent to the original non-associative category (Munch-Maccagnoni, 2014).

Yet, these good properties say very little about how $\mathbf{IMALL}_p$ has been designed. Indeed, it has not been designed initially with the goal of being a good syntax for linear CBPV. To explain its origins, we need to go back to a different work on polarisation than Girard's, to Danos, Joinet

$$
\begin{array}{ll}
\dfrac{\Gamma \vdash t : A \mid \quad \Gamma' \mid e : B \vdash \Delta}{\Gamma, \Gamma' \mid t{\cdot}e : A \multimap B \vdash \Delta}\,(\multimap\vdash) & t_{\varepsilon_1}{\cdot}e_{\varepsilon_2} \overset{\text{def}}{=} \bar{\mu}x^{-}.\left\langle \mu\alpha^{\varepsilon_2}.\left\langle t \,\middle\|\, \bar{\mu}y^{\varepsilon_1}.\langle x \,\|\, y{\cdot}\alpha\rangle^{-}\right\rangle^{\varepsilon_1} \,\middle\|\, e\right\rangle^{\varepsilon_2} \\
\dfrac{\Gamma \vdash t : A \mid \quad \Gamma' \vdash u : B \mid}{\Gamma, \Gamma' \vdash t{\otimes}u : A \otimes B \mid}\,(\vdash\otimes) & t_{\varepsilon_1}{\otimes}u_{\varepsilon_2} \overset{\text{def}}{=} \mu\alpha^{+}.\left\langle t \,\middle\|\, \bar{\mu}x^{\varepsilon_1}.\left\langle u \,\middle\|\, \bar{\mu}y^{\varepsilon_2}.\langle x{\otimes}y \,\|\, \alpha\rangle^{+}\right\rangle^{\varepsilon_2}\right\rangle^{\varepsilon_1} \\
\dfrac{\Gamma \vdash t : A_i \mid}{\Gamma \vdash \iota_i(t) : A_1 \oplus A_2 \mid}\,(\vdash\oplus_i) & \iota_i(t_\varepsilon) \overset{\text{def}}{=} \mu\alpha^{+}.\left\langle t \,\middle\|\, \bar{\mu}x^{\varepsilon}.\langle \iota_i(x) \,\|\, \alpha\rangle^{+}\right\rangle^{\varepsilon} \\
\dfrac{\Gamma \mid e : A_i \vdash \Delta}{\Gamma \mid \pi_i{\cdot}e : A_1 \,\&\, A_2 \vdash \Delta}\,(\&_i\vdash) & (\pi_i{\cdot}e_\varepsilon) \overset{\text{def}}{=} \bar{\mu}x^{-}.\left\langle \mu\alpha^{\varepsilon}.\langle x \,\|\, \pi_i{\cdot}\alpha\rangle^{-} \,\middle\|\, e\right\rangle \\
\dfrac{}{\Gamma \mid \bar{\mu}[\mathrm{dom}(\Gamma, \Delta)] : \mathbf{0} \vdash \Delta}\,(\mathbf{0}\vdash) & \bar{\mu}[\{x_1, \ldots, x_n, \alpha\}] \overset{\text{def}}{=} \bar{\mu}[x_1{\cdot}\ldots{\cdot}x_n{\cdot}\alpha] \\
\dfrac{}{\Gamma \vdash \mu{<}\,\mathrm{dom}\,\Gamma{>} : \top \mid}\,(\vdash\top) & \mu{<}\{x_1, \ldots, x_n\}{>} \overset{\text{def}}{=} \mu{<}x_1{\otimes}\cdots{\otimes}x_n{>}
\end{array}
$$

Figure 4: Derivable rules in $\mathbf{IMALL}_p$.

and Schellinx's work on proof-relevant **LK** (DJS 1997). In this work, DJS investigated various ways of endowing **LK** with a noetherian cut-elimination procedure by means of translations into linear logic. This cut-elimination procedure is also semantics-preserving by construction, for the semantics determined by the translation into linear logic. Using their method, DJS have found a canonical system called $\mathbf{LK}_p^\eta$ whose translation into linear logic adds the fewest modalities ! and ?, and from which other systems could be derived (what we would call $\mathbf{LK}_p$ in the notations of this paper, but which was at the time defined without a term syntax). $\mathbf{LK}_p^\eta$ is also very similar to Girard's original polarised classical sequent calculus, which was designed to have a simple yet proof-relevant denotational semantics (again preserved by cut elimination), and De Morgan identities realized as type isomorphisms. This methodology was applied again using the Curien-Herbelin syntactic technology for sequent calculus (Curien and Herbelin, 2000; Herbelin, 2005) in Munch-Maccagnoni (2009).

The main idea behind DJS's noetherian cut-elimination is to resolve critical pairs in the cut elimination of LK statically, in a direction determined by the type, either towards the "head" $(-)$ or towards the "tail" $(+)$. Annotations $(-)$ or $(+)$ thus correspond to opposite ways of determining priority in a cut:

$$
\begin{aligned}
\langle \mu\alpha^{-}.c \,\|\, \bar{\mu}x^{-}.c' \rangle^{-} &\rhd_{\mathrm{R}} c'[\mu\alpha^{-}.c/x] \\
\langle \mu\alpha^{+}.c \,\|\, \bar{\mu}x^{+}.c' \rangle^{+} &\rhd_{\mathrm{R}} c[\bar{\mu}x^{+}.c/\alpha]
\end{aligned}
$$

In DJS 1997, this annotation is determined by typing;[2] in particular the direction for a specific type is, by design, persistent, in the sense that it cannot change during cut-elimination.

In addition, more constraints appear if we ask for more properties of the system, as with DJS's $\mathbf{LK}_p^\eta$. Mainly, we take as primitive the notions of 1) local reduction of principal cuts, and 2) η-expansion, and we then ask that expansions "play well" with reductions. This has two main consequences:

[2] The first system in DJS 1997, $\mathbf{LK}^{tq}$, has explicit and arbitrary annotations (t, or $-$) and (q, or $+$) on formulæ; this is an equivalent presentation to having a default direction determined by connectives together with "polarity-shifting" unary connectives available to impose a $(-)$ or $(+)$ direction.

- **Polarisation.** We would like that an expansion leaves the reduction order unchanged. Therefore when an expansion is available and seems to force a particular reduction, as in the following example of a cut of type $\oplus$:

$$\begin{aligned}\langle \mu\alpha^{\varepsilon}.c \,\|\, \bar\mu x^{\varepsilon}.c' \rangle^{\varepsilon} &\to_{\mathrm{E}} \langle \mu\alpha^{\varepsilon}.c \,\|\, \bar\mu[x.\langle \iota_1(x) \,\|\, \bar\mu x^{\varepsilon}.c' \rangle \,|\, y.\langle \iota_2(y) \,\|\, \bar\mu x^{\varepsilon}.c' \rangle] \rangle^{\varepsilon} \\ &\rhd_{\mathrm{R}} c[\bar\mu[x.\langle \iota_1(x) \,\|\, \bar\mu x^{\varepsilon}.c' \rangle \,|\, y.\langle \iota_2(y) \,\|\, \bar\mu x^{\varepsilon}.c' \rangle]/\alpha] \\ &\leftarrow^{*}_{\mathrm{E}} c[\bar\mu x^{\varepsilon}.c'/\alpha]\,,\end{aligned}$$

  then this must be the reduction that was determined all along by the annotation:

$$\langle \mu\alpha^{\varepsilon}.c \,\|\, \bar\mu x^{\varepsilon}.c' \rangle^{\varepsilon} \rhd_{\mathrm{R}} c[\bar\mu x^{\varepsilon}.c'/\alpha]$$

  That is to say, $\varepsilon = +$ in this example. Given that the connectives $\mathbf{1}, \otimes, \oplus, \mathbf{0}$ have their expansions on the right-hand side of contexts as above, and the connectives $\multimap, \&, \top$ have their expansion on the left-hand side of expressions, the former are $(+)$ and the latter are $(-)$.

- **Focalisation.** An expansion can reveal reductions as in the following example, again of a cut of type $\oplus$:

$$\begin{aligned}\langle \iota_1(t_+) \,\|\, \bar\mu x^{+}.c \rangle^{+} &\to^{*}_{\mathrm{RE}} \langle \iota_1(t_+) \,\|\, \bar\mu[y.c[\iota_1(y)/x] \,|\, z.c[\iota_2(z)/x]] \rangle^{+} \\ &\to^{*}_{\mathrm{R}} \langle t_+ \,\|\, \bar\mu y^{+}.c[\iota_1(y)/x] \rangle^{+}\end{aligned}$$

  where the reduction in last line is desired because it corresponds to a local reduction of a principal cut for $\oplus$. This is an evidence that $\iota_1(t_+)$ contains a hidden cut, which we require to reveal itself without the intervention of an expansion:

$$\langle \iota_1(t_+) \,\|\, \bar\mu x^{+}.c \rangle^{+} \to^{*}_{\mathrm{R}} \langle t_+ \,\|\, \bar\mu y^{+}.c[\iota_1(y)/x] \rangle^{+}$$

  In $\mathbf{IMALL}_p$, this is obtained by restricting the primitive form of constructors to values and co-values (e.g. $\iota_1(V)$), which forces unrestricted forms (such as $\iota_1(t_+)$) to be derived as in fig. 4:

$$\iota_1(t_+) = \mu\alpha^{+}.\langle t \,\|\, \bar\mu x^{+}.\langle \iota_i(x) \,\|\, \alpha \rangle^{+} \rangle^{+}\,.$$

  In terms of focused (*i.e.* normal) proofs, focalisation is responsible for the atomicity of synchronous phases.

$\mathbf{IMALL}_p$ is obtained by starting from the Curien-Herbelin syntactic toolkit and applying these ideas: in this sense it is a proof-relevant **IMALL** whose theory of noetherian proof reduction and semantics-preserving proof equivalence derives from the principles of polarisation and focalisation.

1.3. Adjunctions in non-associative categories

The notion that reductions and expansions have to "play well" together is traditionally characterised in categorical semantics with the presence of adjunctions, in which connectives are left or right adjoints. One recent realisation is that this characterisation already makes sense in bare deductive systems (non-associative categories). Let us recall the notion of adjunction between bare functors over non-associative categories introduced in MMMM 2026.

Definition 4. A **bare functor** $F : \mathscr{D} \rightharpoonup \mathscr{E}$ between non-associative categories $\mathscr{D}$ and $\mathscr{E}$ is a morphism of the reflexive graphs $\mathscr{D}$ and $\mathscr{E}$, in other words a bare functor $F : \mathscr{D} \rightharpoonup \mathscr{E}$ is a functor that does not necessarily preserve composition. A **proper** functor is one that does (notation $\mathscr{D} \to \mathscr{E}$).

Definition 5. An **adjunction** over non-associative categories $\mathscr{D}$ and $\mathscr{E}$

$$\begin{tikzcd} \mathscr{D} \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp"] & \mathscr{E} \arrow[l, bend left, "G"] \end{tikzcd}$$

consists of two bare functors $F : \mathscr{D} \rightharpoonup \mathscr{E}$ and $G : \mathscr{E} \rightharpoonup \mathscr{D}$ and a family of bijections:

$$\varphi_{A,B} : \mathscr{E}(FA, B) \cong \mathscr{D}(A, GB)$$

natural separately in $A \in \mathscr{D}$ and $B \in \mathscr{E}$.

Note that an adjunction over associative categories is an adjunction in the usual sense, since the bare functors are necessarily proper in that case.

Definition 6. For any pair of non-associative categories $\mathscr{D}, \mathscr{E}$, their **direct product** $\mathscr{D} \times \mathscr{E}$ is the non-associative category with objects $|\mathscr{D}| \times |\mathscr{E}|$ and morphisms $(\mathscr{D} \times \mathscr{E})((A, A'), (B, B')) = \mathscr{D}(A, B) \times \mathscr{E}(A', B')$ defined in the obvious way. For $\mathscr{D}$ a non-associative category, we define the **diagonal functor** $\Delta : \mathscr{D} \rightharpoonup \mathscr{D} \times \mathscr{D}$ with $\Delta A = (A, A)$ and $\Delta f = (f, f)$.

Note that Δ preserves both identities and composition; it is in fact a proper functor.

Proposition 7. *In* $\mathbf{IMALL}_p$ *considered as a non-associative category* $\mathscr{D}$*, the diagonal functor* Δ *has a left adjoint* $\hat{\oplus} : \mathscr{D} \times \mathscr{D} \rightharpoonup \mathscr{D}$ *with* $A \mathbin{\hat{\oplus}} B = A \oplus B$ *on objects, satisfying a family of isomorphisms:*

$$(A \oplus B \vdash C) \cong (A \vdash C) \times (B \vdash C)$$

natural in $(A, B) \in (\mathscr{D} \times \mathscr{D})^{\mathrm{op}}$ *and in* $C \in \mathscr{D}$ *separately.*

Δ *also has a right adjoint* $\hat{\&} : \mathscr{D} \times \mathscr{D} \rightharpoonup \mathscr{D}$ *with* $A \mathbin{\hat{\&}} B = A \& B$ *on objects, satisfying a family of isomorphisms:*

$$(A \vdash B \& C) \cong (A \vdash B) \times (A \vdash C)$$

natural in $A \in \mathscr{D}^{\mathrm{op}}$ *and in* $(B, C) \in \mathscr{D} \times \mathscr{D}$ *separately.*

More details are given in the case of sums in Section B (the case of the product & is symmetric). The relevant structure of $\otimes$ and $\multimap$ is described in MMMM 2026; we omit it in this paper for simplicity.

2. ILL$^\lozenge$: Resource and effect modalities

In this section, we extend the previous sequent calculus for multiplicative additive intuitionistic linear logic **IMALL** with a resource or exponential modality noted $A \mapsto !A$ in order to obtain a sequent calculus for the full propositional intuitionistic linear logic **ILL**, what we write

$$\mathbf{ILL} \quad = \quad \mathbf{IMALL} \; + \; !$$

We then define a sequent calculus for **ILL** combined with an effect modality noted $A \mapsto \lozenge A$ required to be strong with respect to the resource modality, in the sense that it comes with the axiom schema

$$!A \otimes \lozenge B \quad \vdash \quad \lozenge(!A \otimes B)$$

We obtain in that way the intuitionistic modal logic

$$\mathbf{ILL}^\lozenge \quad = \quad \mathbf{ILL} \; + \; \lozenge \quad = \quad \mathbf{IMALL} \; + \; ! \; + \; \lozenge$$

whose sequent calculus is presented in Figure 5. Note that $\mathbf{ILL}^\lozenge$ may be seen as a linear variant of the constructive modal logic **CS4** (Alechina, Mendler et al., 2001) where we find convenient (and in the tradition of linear logic) to write $A \mapsto !\,A$ for the necessity modality $A \mapsto \Box A$.

positive $P, Q, A_+ ::= \cdots \mid !A$

negative $N, M, A_- ::= \cdots \mid \Diamond A$

(a) Types

- $!\Gamma$ stands for a typing context $!A_1, \ldots, !A_n$.

(b) Judgements

$$\frac{\Gamma \vdash B}{\Gamma, !A \vdash B}\,(\mathrm{w}) \qquad \frac{\Gamma, !A, !A \vdash B}{\Gamma, !A \vdash B}\,(\mathrm{c})$$

(c) Structure

$$\frac{!\Gamma \vdash A}{!\Gamma \vdash !A}\,(\vdash !) \qquad \frac{\Gamma, A \vdash B}{\Gamma, !A \vdash B}\,(! \vdash)$$

$$\frac{\Gamma \vdash A}{\Gamma \vdash \Diamond A}\,(\vdash \Diamond) \qquad \frac{!\Gamma, A \vdash \Diamond B}{!\Gamma, \Diamond A \vdash \Diamond B}\,(\Diamond \vdash)$$

(d) Logic

Figure 5: $\mathbf{ILL}^{\Diamond}$ = **IMALL** + the rules above

2.1. Linear CBPV models with resource and effect modalities

As in Proposition 1 for **IMALL**, there is an interpretation of linear CBPV models as deductive systems that validate the rules of Figure 5. Let us first recall that a linear CBPV model with a resource modality (CFMM 2016) is given by a linear CBPV adjunction (1) together with a linear-non-linear adjunction

$$\begin{tikzcd} \mathscr{M} \arrow[r, bend left, "Lin"] \arrow[r, phantom, "\perp"] & \mathscr{V} \arrow[l, bend left, "Mult"] \arrow[r, bend left, "\uparrow"] \arrow[r, phantom, "\perp"] & \mathscr{S} \arrow[l, bend left, "\downarrow"] \end{tikzcd} \tag{2}$$

that is, a symmetric monoidal adjunction as above where the monoidal structure on $\mathscr{M}$ is cartesian, leading to:

Proposition 8 (CFMM 2016)**.** *The deductive system* $\mathbf{dupl}_{\uparrow,\downarrow}$ *associated to a linear CBPV adjunction with resource modality* (2) *satisfies the rules of* **ILL***, that is to say the rules of Figure 5 without the modality* $\Diamond$*.*

Interpreting all the rules from fig. 5 in an analogous of Proposition 8 for of $\mathbf{ILL}^{\Diamond}$ is possible and requires an additional structure in the form of an effect modality: something like a monad $T = G \circ F$ on $\mathscr{S}$ which is strong with respect to an action of $\mathscr{M}^{\mathrm{op}}$ on $\mathscr{S}$ (Melliès, 2012b) leading to a series of "strong" adjunctions

$$\begin{tikzcd} \mathscr{M} \arrow[r, bend left, "Lin"] \arrow[r, phantom, "\perp"] & \mathscr{V} \arrow[l, bend left, "Mult"] \arrow[r, bend left, "\uparrow"] \arrow[r, phantom, "\perp"] & \mathscr{S} \arrow[l, bend left, "\downarrow"] \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp"] & \mathscr{C} \arrow[l, bend left, "G"] \end{tikzcd} \tag{3}$$

In order to make it no more complicated than necessary for our purposes, here with $F \dashv G$ being strong we mean that T is provided with a natural family of morphisms in $\mathscr{V}$,

$$lstr_{X,P} : Lin\, X \otimes \downarrow T \uparrow P \to \downarrow T \uparrow (Lin\, X \otimes P)\,,$$

making four usual diagrams commute—corresponding to the notion of *strength with respect to* $Lin : \mathscr{M} \to \mathscr{V}$ (Blute, Cockett and Seely, 1996; Hasegawa, 2002) for the monad $\downarrow \circ T \circ \uparrow$ on $\mathscr{V}$. This notion might seem somewhat restrictive, and it is going to be all the more interesting that this is enough for our purposes.

Example 9.

- Any (non-polarised) model of intuitionistic linear logic, that is to say a situation (2) in which the symmetric monoidal category $\mathscr{V}$ is closed, $\mathscr{S} = \mathscr{V}$, and $\uparrow = \downarrow = \mathrm{Id}_{\mathscr{V}}$, has a situation (3) in which T is the exception monad $\mathcal{E} = - \oplus E$ (for any $E \in \mathscr{V}$), which is strong with respect to *Lin*, in the sense that the linear tautologies

$$!\Gamma \otimes (A \oplus E) \vdash (!\Gamma \otimes A) \oplus E$$

underlie a natural family of morphisms

$$\mathit{Lin}\, X \otimes \mathcal{E}A \to \mathcal{E}(\mathit{Lin}\, X \otimes A)$$

satisfying the four coherence laws. Diagrammatically:

$$\begin{tikzcd} \mathscr{M} \arrow[r, bend left, "\mathit{Lin}"] \arrow[r, phantom, "\perp"] & \mathscr{V} \arrow[l, bend left, "\mathit{Mult}"] \arrow[r, bend left, "F^{\mathcal{E}}"] \arrow[r, phantom, "\perp"] & \mathscr{V}^{\mathcal{E}} \arrow[l, bend left, "G^{\mathcal{E}}"] \end{tikzcd}$$

- Any model of linear logic, that is to say (2) in which $\mathscr{V}$ is $*$-autonomous, $\mathscr{S} = \mathscr{V}$, and $\uparrow = \downarrow = \mathrm{Id}$, has a situation (3) in which $T = ?$ is the dual of $!$, $\mathscr{C} = \mathscr{M}^{\mathrm{op}}$, $F = \mathit{Mult} \circ \neg$ and $G = \neg \circ \mathit{Lin}$, and in which the linear tautologies

$$!\Gamma \otimes ?A \vdash ?(!\Gamma \otimes A)$$

underlie again a strength of ? in the previous sense.

$$\begin{tikzcd} \mathscr{M} \arrow[r, bend left, "\mathit{Lin}"] \arrow[r, phantom, "\perp"] & \mathscr{V} \arrow[l, bend left, "\mathit{Mult}"] \arrow[r, bend left, "\mathit{Mult} \circ \neg"] \arrow[r, phantom, "\perp"] & \mathscr{M}^{\mathrm{op}} \arrow[l, bend left, "\neg \circ \mathit{Lin}"] \end{tikzcd}$$

- By not assuming $\mathscr{V}$ to be $*$-autonomous but merely a dialogue category (a symmetric monoidal category having a negation functor $\neg : \mathscr{V} \to \mathscr{V}^{\mathrm{op}}$, in the sense of having an exponentiable object), we arrive at a linear CBPV model with a resource modality (2) with $\mathscr{S} = \mathscr{V}^{\mathrm{op}}$ and with the adjunction of negation with itself. Such models give again rise to $\mathbf{ILL}_p^{\Diamond}$ models (3) with $\mathscr{C} = \mathscr{M}^{\mathrm{op}}$, $F = \mathit{Mult}$ and $G = \mathit{Lin}$.

$$\begin{tikzcd} \mathscr{M} \arrow[r, bend left, "\mathit{Lin}"] \arrow[r, phantom, "\perp"] & \mathscr{V} \arrow[l, bend left, "\mathit{Mult}"] \arrow[r, bend left, "\neg"] \arrow[r, phantom, "\perp"] & \mathscr{V}^{\mathrm{op}} \arrow[l, bend left, "\neg"] \arrow[r, bend left, "\mathit{Mult}"] \arrow[r, phantom, "\perp"] & \mathscr{M}^{\mathrm{op}} \arrow[l, bend left, "\mathit{Lin}"] \end{tikzcd} \tag{4}$$

This situation has been used to explain translations of (polarised) classical logic into linear logic and to describe game models of linear logic in Melliès and Tabareau (2010).

Proposition 10. *The deductive system* $\mathbf{dupl}_{\uparrow,\downarrow}$ *associated to a linear CBPV adjunction with resource and effect modalities* (3) *satisfies the rules of* $\mathbf{ILL}^{\Diamond}$ *of Figure 5.*

Up until now, we did not detail the interpretation of formulæ of **IMALL** as objects of linear CBPV models since it was rather straightforward. However, the interpretation is no longer straightforward for ! and $\Diamond$. Indeed, the valid interpretation of the rules for ! and $\Diamond$ require an extra polarity shift:

$$\begin{aligned} !A &= \mathit{Lin} \circ \mathit{Mult} \circ {\downarrow}(A^{-}) \\ \Diamond A &= T \circ {\uparrow}(A^{+}) \end{aligned} \tag{5}$$

This phenomenon corresponds to the fact that starting from an oblique morphism

$$\mathscr{O}(\mathit{Lin}\, X_1 \otimes \cdots \otimes \mathit{Lin}\, X_n, {\uparrow} P)$$

one cannot in general obtain an oblique morphism

$$\mathscr{O}(\mathit{Lin}\, X_1 \otimes \cdots \otimes \mathit{Lin}\, X_n, {\uparrow}\, \mathit{Lin}\, \mathit{Mult}\, P)\,,$$

merely one in

$$\mathscr{O}(\mathit{Lin}\, X_1 \otimes \cdots \otimes \mathit{Lin}\, X_n, {\uparrow}\, \mathit{Lin}\, \mathit{Mult}\, {\downarrow}{\uparrow} P)\,. \tag{6}$$

This phenomenon was described semantically in Curien, Fiore and Munch-Maccagnoni (CFMM 2016) but was observed syntactically as early as Andreoli (1992) with the facts that the introduction of ! terminates a synchronous phase and that the introduction of ? (i.e. ! on the left) terminates an asynchronous phase in focusing proof search. The monad T undergoes a dual phenomenon. This has consequences on the completeness of the interpretation as we will see.

2.2. The calculus $\mathbf{ILL}_p^{\Diamond}$: A polarised syntax for ! and $\Diamond$ modalities

The calculus $\mathbf{ILL}_p^{\Diamond}$ is defined in Figure 6. It is $\mathbf{IMALL}_p$ with modalities ! and $\Diamond$ following a syntax introduced for linear logic exponentials !, ? in Munch-Maccagnoni (2009). One of the goals of this paper is to explain this somewhat unintuitive, but mathematically justified syntax.

The calculus $\mathbf{ILL}_p^{\Diamond}$ enjoys the same good properties we mentioned for $\mathbf{IMALL}_p$: its type system expresses the logic $\mathbf{ILL}^{\Diamond}$, and its derivations can be soundly interpreted in any linear CBPV model with a resource and an effect modalities. The reduction relation $\to_{\mathrm{R}}$ defines again a noetherian and semantics-preserving cut-elimination strategy for $\mathbf{ILL}^{\Diamond}$. The $\to_{\mathrm{R}}$-normal and $\to_{\mathrm{E}}$-long terms are again in correspondence with standard focused normal forms, from which the completeness of focusing proof search follows. (The interested reader can consult Munch-Maccagnoni (2017) for the case of **ILL** treated in detail.)

However, we lose the following completeness property: it does not seem possible to recover a resource modality on the category of thunkables nor an effect modality on the category of linears. As we will see this syntax corresponds in fact to linear CBPV with a resource comonad relative to $\downarrow$ and a monad relative to $\uparrow$, strong with respect to the comonad, as defined later.

Now we turn to the focalisation and polarisation properties that gave rise to this syntax—indeed, the initial goal was not to provide a syntax for monads and comonads relative to a CBPV adjunction! The first observation is that the η-rules are very different for ! and $\Diamond$ as compared to other connectives. In terms of mere provability, none of their left- and right-introduction rules are invertible. There is a weaker property of "semi-invertibility" observed in Laurent (2002): provided that the context Γ is of the right form, the modality ! is invertible on the right (in terms of mere provability). We adopt a slightly weaker notion of invertibility (or η-expansion) for ! and $\Diamond$, which is valid semantically, relying on the following principle: a well-typed value of type $!A$ must come from either a promotion or a variable (possibly with some structural rules applied); intuitively it corresponds to a "box" in terms of proof nets for linear logic. In particular the context of this

$$
\begin{array}{lrclcl}
 & & & & ! & \lozenge \\
\text{Values:} & V, W & ::= & \dots & \mu!\alpha.c & \lozenge V \\
\text{Stacks:} & S & ::= & \dots & !S & \tilde{\mu}\lozenge x.c
\end{array}
$$

(a) Terms

$$
\begin{array}{lccl}
(R!): & \langle \mu!\alpha.c \,\|\, !S \rangle^{+} & \triangleright_{\mathrm{R}} & c[S/\alpha] \\
(R\lozenge): & \langle \lozenge V \,\|\, \tilde{\mu}\lozenge x.c \rangle^{-} & \triangleright_{\mathrm{R}} & c[V/x]
\end{array}
$$

(b) Reduction rules

$$
\begin{array}{lccl}
(E!): & V & \triangleright_{\mathrm{E}} & \mu!\alpha.\langle V \,\|\, !\alpha \rangle^{+} \\
(E\lozenge): & S & \triangleright_{\mathrm{E}} & \tilde{\mu}\lozenge x.\langle \lozenge x \,\|\, S \rangle^{+}
\end{array}
$$

(c) Extensionality rules

- $!\Gamma$ stands for a typing context $x_1 : !A_1, \dots, x_n : !A_n$.
- $\lozenge\Delta$ stands for a typing context $\alpha : \lozenge A$.

(d) Judgements

- $\mathbf{\Sigma^{!}(\Gamma;\Gamma')}$ is the subset of $\Sigma(\Gamma;\Gamma')$ of maps that are bijective on variables not of the form $!A$.
- The structural rules for $\mathbf{ILL}_p^{\lozenge}$ are given by taking $\Sigma^{!}(\Gamma;\Gamma')$ for the range of structural maps (renaming and exchange, as well as weakening and exchange for formulae of the form $!A$).

(e) Structure

$$
\frac{c : (!\Gamma \vdash \alpha : A)}{!\Gamma \vdash \mu!\alpha.c : !A \mid}\,(\vdash !)
\qquad
\frac{\Gamma \mid S : A \vdash \Delta}{\Gamma \mid !S : !A \vdash \Delta}\,(! \vdash_f)
$$

$$
\frac{\Gamma \vdash V : A}{\Gamma \vdash \lozenge V : \lozenge A}\,(\vdash_f \lozenge)
\qquad
\frac{c : (!\Gamma \mid x : A \vdash \lozenge\Delta)}{!\Gamma \mid \tilde{\mu}\lozenge x.c : \lozenge A \vdash \lozenge\Delta}\,(\lozenge \vdash)
$$

(f) Logic

$$
\frac{\Gamma \mid e : A \vdash \Delta}{\Gamma \mid !e : !A \vdash \Delta}\,(! \vdash)
\qquad
!e_{\varepsilon} \stackrel{\text{def}}{=} \tilde{\mu}x^{+}.\langle \mu\alpha^{\varepsilon}.\langle x \,\|\, !\alpha \rangle \,\|\, e \rangle
$$

$$
\frac{\Gamma \vdash t : A}{\Gamma \vdash \lozenge t : \lozenge A}\,(\vdash \lozenge)
\qquad
\lozenge t_{\varepsilon} \stackrel{\text{def}}{=} \mu\alpha^{-}.\langle t \,\|\, \tilde{\mu}x^{\varepsilon}.\langle \lozenge\alpha \,\|\, x \rangle \rangle
$$

(g) Additional derived rules in $\mathbf{ILL}_p^{\lozenge}$.

Figure 6: $\mathbf{ILL}_p^{\lozenge} = \mathbf{IMALL}_p$ + above

value is of the form $!\Gamma$. The weak invertibility principle for ! is then reflected in the syntax with a promotion constructor in the shape of pattern matching together with a dereliction constructor in the shape of a pattern:

$$!\Gamma \vdash \mu!\alpha.c \qquad \Gamma \mid !e : !A \vdash \Delta$$

in such a way that the corresponding expansion correctly reflects a (weak) invertibility of ! on the right:

$$V \triangleright_{\mathrm{E}} \mu!\alpha.\langle V \,\|\, !\alpha \rangle^{+} \tag{7}$$

and that it is well-typed. Dually, a well-typed co-value of type $\lozenge A$ must come from either an extension or a covariable, such that its context is of the form $!\Gamma, \lozenge\Delta$, in such a way that the expansion

$$S \triangleright_{\mathrm{E}} \tilde{\mu}\lozenge x.\langle \lozenge x \,\|\, S \rangle^{+}$$

is well-typed.[3] Starting from these desired η-expansions, polarisation and focalisation again help us design the rest of the calculus.

- **Polarisation.** The weak invertibility property is not sufficient to force ! to be a negative connective (and dually to force $\lozenge$ to be a positive connective) since they apply conditionally. On the contrary ! needs to be positive and $\lozenge$ negative to ensure that values of type $!A$ and covalues of type $\lozenge A$ are boxes (promotions or variables for !, extension or covariable for $\lozenge$). Indeed, in a generic cut

$$\langle t \,||\, \tilde{\mu}x^{\varepsilon}.c\rangle^{\varepsilon}$$

where t is an expression of type $!A$, and where x might occur several or zero times in c', we do not know that t can be duplicated or erased (e.g. corresponds to a box, if we have linear logic in mind). In general this might even be ill-typed since the context of t is not necessarily of the form $!\Gamma$. By assigning the polarity $\varepsilon = +$ to the modality !, we enforce a call-by-value reduction:

$$\begin{aligned} \text{case } t = \mu\alpha^{+}.c' : \quad & \langle \mu\alpha^{+}.c' \,||\, \tilde{\mu}x^{+}.c\rangle^{+} \rhd_{\mathrm{R}} c'[\tilde{\mu}x^{+}.c/\alpha] \\ \text{case } t = V \neq \mu\alpha^{+}.c' : \quad & \langle V \,||\, \tilde{\mu}x^{+}.c\rangle^{+} \rhd_{\mathrm{R}} c[V/\alpha] \end{aligned}$$

which has the crucial consequence that duplication and erasure (and expansion (7)) can only happen for V of the form $V = x$ or $V = \mu!\alpha.c$. Dually, assigning the polarity $\varepsilon = -$ to the modality $\lozenge$ enforces a call-by-name reduction meaning that a covalue S of type $\lozenge A$ is necessarily some $S = \tilde{\mu}\lozenge x.c$ or some $S = \alpha$.

- **Focalisation.** As we have seen, the focalisation properties which apply for the positive connectives $\oplus$ and $\otimes$ on the right (and for the negative connectives $\multimap$ and & on the left) do not apply for ! on the right (and $\lozenge$ on the left). This corresponds semantically to the observation on oblique morphisms (6) from the previous section where extra shifts were necessary. However, (weak) focalisation properties are available for ! on the left (dereliction) and $\lozenge$ on the right (return), and are in fact forced by weak invertibility on the opposite side:

$$\begin{aligned} \langle V \,||\, !e_{-}\rangle^{+} &\to_{\mathrm{E}} \langle \mu!\alpha.\langle V \,||\, !\alpha\rangle^{+} \,||\, !e_{-}\rangle^{+} \\ &\to^{*}_{\mathrm{R}} \langle \mu\alpha^{-}.\langle V \,||\, !\alpha\rangle^{+} \,||\, e_{-}\rangle^{-} \\ \langle \lozenge t_{+} \,||\, S\rangle^{-} &\to_{\mathrm{E}} \langle \lozenge t_{+} \,||\, \tilde{\mu}\lozenge x.\langle \lozenge x \,||\, S\rangle^{-}\rangle^{-} \\ &\to^{*}_{\mathrm{R}} \langle t_{+} \,||\, \tilde{\mu}x^{+}.\langle \lozenge x \,||\, S\rangle^{-}\rangle^{+} \end{aligned}$$

The weak focalisations implementing the derived equivalences above for dereliction and return are given in fig. 6g:

$$\begin{aligned} \langle V \,||\, !e_{-}\rangle^{+} &\to_{\mathrm{R}} \langle \mu\alpha^{-}.\langle V \,||\, !\alpha\rangle^{+} \,||\, e_{-}\rangle^{-} \\ \langle \lozenge t_{+} \,||\, S\rangle^{-} &\to_{\mathrm{R}} \langle t_{+} \,||\, \tilde{\mu}x^{+}.\langle \lozenge x \,||\, S\rangle^{-}\rangle^{+} \end{aligned}$$

These focalisation properties are weak in the sense that the following unrestricted focalisations are unsound in general:

$$\begin{aligned} \langle t_{+} \,||\, !e_{-}\rangle^{+} &\to_{(\text{unsound})} \langle \mu\alpha^{-}.\langle t_{+} \,||\, !\alpha\rangle^{+} \,||\, e_{-}\rangle^{-} \\ \langle \lozenge t_{+} \,||\, e_{-}\rangle^{-} &\to_{(\text{unsound})} \langle t_{+} \,||\, \tilde{\mu}x^{+}.\langle \lozenge x \,||\, e_{-}\rangle^{-}\rangle^{+}\,. \end{aligned}$$

[3] and whose interpretations into linear CBPV models are sound due to a stronger observation: the interpretation of expressions of type $!A$ that are values are !-coalgebra morphisms whereas the interpretation of coexpressions of type $\lozenge A$ that are covalues are $\lozenge$-algebra morphisms, although the initial motivations were syntactic (in terms of proof net boxes for ! and ? in linear logic, as explained).

The calculus $\mathbf{ILL}_p^{\lozenge}$ is $\mathbf{IMALL}_p$ augmented with a resource modality ! and an effect modality $\lozenge$, for which the earlier recipe no longer applies. We applied the specific and delicate treatment for the modalities described above. This treatment still stems from considerations about polarisation and focalisation, guided again from the idea of valid reductions and expansions playing well together. The resulting syntaxes for ! and $\lozenge$ are arguably more complicated, but are justified by a correspondence with focusing proof search, as previously mentioned, and a sound interpretation into call-by-push-value models with resource and effect modalities. We now seek to recover the completeness property which was missing from CFMM 2016 by moving to relative (co)monads.

2.3. Linear call-by-push-value models and relative resource and effect modalities

We now define relative monads and comonads, via relative adjunctions and coadjunctions (Altenkirch, Chapman and Uustalu, 2015; Arkor and McDermott, 2024). They were previously used for modelling resource and effect modalities over CBPV models by Melliès (2025) and in Jiang, Xue and New (2025), respectively.

Definition 11 (Relative adjunctions and coadjunctions on categories)**.** Let $\mathscr{A}$ and $\mathscr{B}$ be two categories and $L : \mathscr{A} \to \mathscr{B}$ be a functor. An ***L*-relative adjunction** $F \dashv_L G$ consists of a category $\mathscr{C}$, two functors $F : \mathscr{A} \to \mathscr{C}$ and $G : \mathscr{C} \to \mathscr{B}$, and a family of bijections:

$$\Phi_{A,C} : \mathscr{C}(FA, C) \cong \mathscr{B}(LA, GC).$$

natural in A and C. Diagrammatically:

$$\begin{tikzcd} & \mathscr{C} \arrow[dr, bend left, "G"] \arrow[d, phantom, "\dashv"] & \\ \mathscr{A} \arrow[ur, bend left, "F"] \arrow[rr, "L"'] & {} & \mathscr{B} \end{tikzcd} \tag{8}$$

We note $\eta : L \to G \circ F$ the unit of the adjunction.

Dually, for a functor $R : \mathscr{B} \to \mathscr{A}$, an ***R*-relative coadjunction** $F \;{}_R\!\dashv G$ consists of a category $\mathscr{C}$, two functors $F : \mathscr{C} \to \mathscr{A}$ and $G : \mathscr{B} \to \mathscr{C}$, and a family of bijections:

$$\Psi_{C,B} : \mathscr{A}(FC, RB) \cong \mathscr{C}(C, GB).$$

natural in B and C. Diagrammatically:

$$\begin{tikzcd} & \mathscr{C} \arrow[dl, bend right, "F"'] \arrow[d, phantom, "\dashv"] & \\ \mathscr{A} & {} & \mathscr{B} \arrow[ll, "R"] \arrow[ul, bend right, "G"'] \end{tikzcd} \tag{9}$$

We note $\varepsilon : F \circ G \to R$ the counit of the coadjunction.

We can now revisit the interpretation from Section 2.1 in light of the notions of relative adjunction and coadjunction starting from the following simple fact about relative (co)adjunctions:

Proposition 12. *In* (5) *one has a* $\uparrow$*-relative coadjunction* $Lin \;{}_{\uparrow}\!\dashv Mult \circ \uparrow$*, and dually, adjoint resolutions* $F \dashv G$ *of a comonad* T *on* $\mathscr{S}$ *give rise to* $\downarrow$*-relative adjunctions* $F \circ \downarrow \dashv_{\downarrow} G$*.*

We observe that our interpretation of $\mathbf{ILL}^{\lozenge}$ into linear CBPV models (5) makes only use of this relative structure. Indeed, assume we are given on top of a linear CBPV adjunction (1)

- a $\uparrow$-relative adjunction

$$Lin \;{}_{\uparrow}\!\dashv Mult : \mathscr{S} \to \mathscr{M}$$

  where $Lin : \mathscr{M} \to \mathscr{V}$ is strong monoidal, defining a relative symmetric monoidal comonad $! = Lin \circ Mult$, and

- a $\downarrow$-relative adjunction

$$F \dashv_{\downarrow} G : \mathscr{V} \to \mathscr{C}$$

  defining a relative monad $\Diamond = G \circ F$, together with a strength with respect to Lin, in the form of a natural family of morphisms

$$str_{X,P} : Lin\, X \otimes \downarrow\!\Diamond P \to \downarrow\!\Diamond(Lin\, X \otimes P) \tag{10}$$

  making the four usual diagrams commute.

Diagrammatically:

$$\begin{tikzcd}
\mathscr{M} \arrow[r, bend left, "Lin"] \arrow[r, phantom, "\perp"] & \mathscr{V} \arrow[r, bend left, "\uparrow"] \arrow[r, phantom, "\perp"] \arrow[rr, bend left=50, "F"] & \mathscr{S} \arrow[l, bend left, "\downarrow"] \arrow[ll, bend left=50, "Mult"] \arrow[r, phantom, "\perp"] & \mathscr{C} \arrow[l, bend left, "G"]
\end{tikzcd} \tag{11}$$

Then one has:

Proposition 13. *The deductive system* $\mathbf{dupl}_{\uparrow,\downarrow}$ *associated to a linear CBPV adjunction with resource and effect modalities in the relative sense of* (11) *satisfies the rules of* $\mathbf{ILL}^{\Diamond}$.

Indeed, rephrasing in terms of the oblique morphism distributor, we have a natural family of bijections:

$$\mathscr{O}(!A^-, B^-) \cong \mathscr{M}(Mult\, A^-, Mult\, B^-) \tag{12}$$

mapping into $\mathscr{O}(!A^-, (!B)^-)$, thus corresponding to the rule $(\vdash\, !)$ (with a single antecedent, omitting the lax structure at the moment). Symmetrically for $\Diamond$, we have a natural family of bijections:

$$\mathscr{O}(A^+, \Diamond B^+) \cong \mathscr{C}(FA^+, FB^+) \tag{13}$$

mapping into $\mathscr{O}((\Diamond A)^+, \Diamond B^+)$, thus corresponding to the rule $(\Diamond \vdash)$ (again with a single antecedent, omitting the strength).

In order to be more precise, we will now make use of the notions of (co)algebras for relative (co)monads. Just as for their non-relative counterparts, categories of algebras are terminal in a category of adjoint resolutions for a given relative (co)monad, in such a way that for any relative (co)adjunction there exists a comparison functor factoring the right (resp. left) adjoint (Altenkirch, Chapman and Uustalu, 2015, Theorem 2.12). The sets (12) thus map via the comparison functor into the set of !-coalgebra morphisms $\mathscr{V}^!(F^!A^-, F^!B^-)$, in which values of type $!A$ are interpreted, where objects $F^!X^-$ are identified with the types $!X$ of $\mathbf{ILL}_p^{\Diamond}$. Symmetrically, the sets (13) map via the comparison functor into the set of $\Diamond$-algebra morphisms $\mathscr{S}^{\Diamond}(G^{\Diamond}A^+, G^{\Diamond}B^+)$ in which covalues of type $\Diamond A$ are interpreted, where objects $G^{\Diamond}X^+$ are identified with the types $\Diamond X$ of $\mathbf{ILL}_p^{\Diamond}$. The rest of the interpretation of $\mathbf{ILL}_p^{\Diamond}$ into linear CBPV models with resource and effect modalities remains unchanged. Thus:

Proposition 14. *The interpretation of* $\mathbf{ILL}^{\Diamond}$ *from Proposition 13 extends into a sound interpretation of* $\mathbf{ILL}_p^{\Diamond}$.

Definition 15. We call the data (11) given previously an $\mathbf{ILL}_p^{\Diamond}$ *model.*

Examples By Proposition 12, any linear CBPV adjunction with resource and effect modalities (3) provides an $\mathbf{ILL}_p^{\Diamond}$ model. Two series of works, by Jiang, Xue and New (2025) and by Melliès (2025, 2026), motivated a relaxation of the notions of effect and resource modalities to monads relative to $\uparrow$ and to comonads relative to $\downarrow$.

Jiang, Xue and New (2025) investigate an interface for user-defined programming effects in CBPV using Church encodings and coinductive types. Since universal quantification and coinductive types are typically negative, this leads to several examples of monads whose type of monadic computations is negative, including various examples of implementations of exception, continuation, state and free monads. The situations they study amount to particular $\mathbf{ILL}_p^{\Diamond}$ models (11) with $\mathscr{M} = \mathscr{V}$ and $\mathscr{C} = \mathscr{S}^{\Diamond}$, i.e. non-linear CBPV models with an effect modality:

$$\begin{tikzcd}
\mathscr{M} \arrow[r, bend left, "\uparrow"] \arrow[r, phantom, "\perp"] \arrow[rr, bend left=50, "F^{\Diamond}"] & \mathscr{S} \arrow[l, bend left, "\downarrow"] \arrow[r, phantom, "\perp"] & \mathscr{S}^{\Diamond} \arrow[l, bend left, "G^{\Diamond}"]
\end{tikzcd} \tag{14}$$

Melliès (2025, 2026) proposed to extend tensorial logic and dialogue categories to linear logic exponentials, using relative rather than non-relative symmetric monoidal comonads as a way of circumventing obstacles encountered within the programme of *functorial game semantics* (Melliès, 2012a) that are similar to the difficulties we have described previously regarding notions of polarities and focusing for exponentials. This setting corresponds to a variant of the situation (4) involving a dialogue category, now with relative comonad and monad:

$$\begin{tikzcd}
\mathscr{M} \arrow[r, bend left, "Lin"] \arrow[r, phantom, "\perp"] & \mathscr{V} \arrow[r, bend left, "\neg"] \arrow[r, phantom, "\perp"] \arrow[rr, bend left=50, "Mult"] & \mathscr{V}^{\mathrm{op}} \arrow[l, bend left, "\neg"] \arrow[ll, bend left=50, "Mult"] \arrow[r, phantom, "\perp"] & \mathscr{M}^{\mathrm{op}} \arrow[l, bend left, "Lin"]
\end{tikzcd}$$

Another approach to a notion of strength for a relative monad is to assume a given monoidal structure on the target category $\mathscr{S}$, which then allows us to define the extension of the relative monad under a context $\Gamma \in \mathscr{S}$,

$$\mathscr{S}(\Gamma \otimes \uparrow P, \Diamond Q) \to \mathscr{S}(\Gamma \otimes \Diamond P, \Diamond Q) \tag{15}$$

This is an assumption made in Liell-Cock, Shirazi and Staton (2026). As a way of comparison, we assume that $\uparrow$ has a right adjoint $\downarrow$, which is essential in making our definition of strength (10) work; one advantage in return is that it does not depend on having a monoidal structure on $\mathscr{S}$. Such a (strong) monoidal structure on negative types is indeed unavailable in certain situations that interest us, such as with certain categories of algebras. An RMM-LNL model (Liell-Cock, Shirazi and Staton, 2026) is given by a linear-non-linear (LNL) model $F : \mathscr{M} \rightleftarrows \mathscr{S} : G$ together with a strong F-relative monad $\Diamond$ in the above sense (15), and some strong monoidal functor from a monoidal category into $\mathscr{S}$. Further assuming $\mathscr{M}$ is distributive, this is again an instance of a non-linear CBPV model with an effect modality (14). Indeed, recall that an LNL model (Benton, 1994), which consists in a symmetric monoidal adjunction $F : \mathscr{M} \rightleftarrows \mathscr{S} : G$ between a cartesian closed category $\mathscr{M}$ and an symmetric monoidal closed category $\mathscr{S}$, is in particular a (non-linear) CBPV model where $\uparrow = F$ and $\downarrow = G$ (except without sums). In an RMM-LNL model, the natural family of morphisms (10) constituting the strength of the $\Uparrow$-relative monad, instantiated as $lstr_{\Gamma,P} \in \mathscr{M}(\Gamma \times G\Diamond P, G\Diamond(\Gamma \times P))$, derives from

$$\mathcal{S}(F(\Gamma \times P), \Diamond(\Gamma \times P)) \cong \mathcal{S}(F\Gamma \otimes FP, \Diamond(\Gamma \times P)) \to \mathcal{S}(F\Gamma \otimes \Diamond P, \Diamond(\Gamma \times P)) \to$$
$$\mathcal{S}(F\Gamma \otimes FG\Diamond P, \Diamond(\Gamma \times P)) \cong \mathscr{M}(F(\Gamma \times G\Diamond P), \Diamond(\Gamma \times P)) \cong \mathscr{M}(\Gamma \times G\Diamond P, G\Diamond(\Gamma \times P)).$$

Lastly, by moving from (co)adjunctions to relative (co)adjunctions for modelling resource and effect modalities, we indeed recover the missing completeness property of the interpretation, giving rise to the syntactic $\mathbf{ILL}_p^{\Diamond}$ model:

Proposition 16. *The syntactic linear CBPV model $\mathscr{P}_t \rightleftarrows \mathscr{N}_l$ of thunkable and linear terms of $\mathbf{ILL}_p^{\Diamond}$ has a relative comonad $! : \mathscr{N}_l \to \mathscr{P}_t$ on $\Downarrow$, and a relative monad $\Diamond : \mathscr{P}_t \to \mathscr{N}_l$ on $\Uparrow$, given by the corresponding rules of $!$ and $\Diamond$.*

The idea is again that thunkable terms $!N \vdash \Downarrow M$ in the syntactic model are in correspondence to generic terms of type $!N \vdash M$, in such a way that the co-unit matches the dereliction restricted to covalues,

$$\mid \, !\alpha : !N \vdash \alpha : N$$

and the coextension matches promotion, which indeed builds a thunkable morphism:

$$\frac{c : (!N \vdash \alpha : M)}{!N \vdash \mu!\alpha.c : !M \mid}$$

For more details consult Appendix C. It is another sign and testimony of the fundamental nature of linear-non-linear relative coadjunctions introduced in the polarised setting of tensorial logic and functorial game semantics that they also provide a way to resolve a completeness issue appearing in the syntax and semantics of linear CBPV with resource modalities (CFMM 2016).

This last, syntactic, example extends to a more general phenomenon giving rise to a final example of $\mathbf{ILL}_p^{\Diamond}$ models, obtained by completing another $\mathbf{ILL}_p^{\Diamond}$ model with its thunkable and linear morphisms. Remember that, in line with the relationships between direct and indirect models of effectful computation (Führmann, 1999; Selinger, 2001), any linear CBPV model $\uparrow : \mathscr{V} \rightleftarrows \mathcal{S} : \downarrow$ (1) gives rise to a linear CBPV model

$$\begin{tikzcd} \mathscr{P}_t \arrow[rr, bend left, "\Uparrow"] \arrow[rr, phantom, "\perp"] & & \mathscr{N}_l \arrow[ll, bend left, "\Downarrow"] \end{tikzcd} \tag{16}$$

obtained starting from the non-associative category $\mathbf{dupl}_{\uparrow,\downarrow}$ and restricting to positive thunkable and negative linear morphisms. Remember also that this construction is a reflection for the category of adjunctions $\downarrow \dashv \uparrow$, and more precisely a completion (of morphisms of $\mathscr{V}$ into thunkable morphisms, and of morphisms of $\mathcal{S}$ into linear morphisms of the adjunction) provided the starting adjunction has its unit mono and its counit epi (Munch-Maccagnoni, 2014). Then, resource and effect modalities on $\mathscr{V} \rightleftarrows \mathcal{S}$, whether non-relative (3) or relative (11), give rise to relative resource and effect modalities on $\mathscr{P}_t \rightleftarrows \mathscr{N}_l$. This is a corollary to the results established in the next section (Corollary 35). In the non-relative case, however, the same obstacle (6) previously mentioned prevents us from obtaining (directly and in general) a non-relative comonad on $\mathscr{P}_t$ and a non-relative monad on $\mathscr{N}_l$.

These results motivate in the rest of the paper an investigation from the non-associative perspective into broader notions of model based on resource and effect modalities relative to the linear CBPV adjunction.

3. Adjunctions over duploids and relative adjunctions

3.1. Prelude: distributive closed duploids and distributive dialogue duploids

We have recalled previously that every adjunction $\uparrow \dashv \downarrow$ induces a non-associative category $\mathbf{dupl}_{\uparrow,\downarrow}$; it is in fact a duploid as defined next.

Definition 17. An object A of a non-associative category is **positive** if every morphism $f : A \to B$ is linear. It is **negative** if every morphism $f : B \to A$ is thunkable. A **duploid** is a non-associative category in which every object is positive or negative (or both), and in which the identity functor has both a positive left adjoint (which we write $\Downarrow$) and a negative right adjoint (written $\Uparrow$). Diagrammatically:

$$\begin{tikzcd} \mathscr{D} \arrow[r, bend left, "\Downarrow"] \arrow[r, phantom, "\perp"] & \mathscr{D} \arrow[l, bend left, "\Uparrow"] \end{tikzcd}$$

Definition 18. An adjunction between two non-associative categories $F : \mathscr{D} \rightleftharpoons \mathscr{E} : G$ (Definition 5) is said to be **positive** when the image of F on objects is positive. Dually, it is **negative** when the image of G on objects is negative.

The notions of *symmetric monoidal closed duploid* and of *dialogue duploid*, providing non-associative categorical and axiomatic semantics to **IMLL** (intuitionistic multiplicative linear logic) and **MLL** (classical multiplicative linear logic) respectively, are defined in MMMM 2026. These non-associative categorical and axiomatic semantics are designed to ensure that their equational theory and normal forms coincide with the syntax of the multiplicative fragment of the corresponding L-calculi.

We now define here the distributive structure in the continuation of Section 1.3.

Definition 19. Let $\mathscr{D}$ be a non-associative category. **Finite sums** on $\mathscr{D}$ is the structure given by: 1) a positive left adjoint to the diagonal functor $\Delta : \mathscr{D} \to \mathscr{D} \times \mathscr{D}$, which we write $\oplus$, 2) a positive left adjoint to the functor $\mathscr{D} \to *$ into the trivial category, in other words a positive initial object, which we write $\mathbf{0}$. **Finite products** (notation $\&, \top$) is the structure given by finite sums on $\mathscr{D}^{\mathrm{op}}$.

Definition 20. Let $\mathscr{D}$ be a symmetric monoidal duploid. $\mathscr{D}$ is **distributive** if it has finite sums and if for all $A, B, C \in \mathscr{D}$ the canonical thunkable map

$$(A \otimes B) \oplus (A \otimes C) \to A \otimes (B \oplus C)$$

has a thunkable inverse.

3.2. A correspondence between relative adjunctions and adjunctions in non-associative direct semantics

Our goal is to extend this non-associative categorical semantics by introducing a notion of linear-non-linear adjunction and strong monad for duploids. One main observation of the present paper is that the appropriate generalisation of linear-non-linear adjunctions to a polarised setting is in correspondence with the notion of relative coadjunction with respect to the polarity shift $\Downarrow$ appearing as right adjoint in the adjunction associated to the duploid

$$\begin{tikzcd} \mathscr{P}_t \arrow[r, bend left, "\Uparrow"] \arrow[r, phantom, "\perp"] & \mathscr{N}_l \arrow[l, bend left, "\Downarrow"] \end{tikzcd}$$

between the category of positive thunkable maps and the category of negative linear maps, whereas the strong monads on duploids are in correspondence with the notion of relative adjunction with respect to the adjoint polarity shift $\Uparrow$.

We first explain how, given an adjunction $L \dashv R$, an R-relative coadjunction $F\ {}_R\!\dashv G$ induces a positive adjunction on the duploid $\mathbf{dupl}_{L,R}$ (Proposition 25). By duality, this implies that every L-relative adjunction $F \dashv_L G$ induces a negative adjunction on the duploid $\mathbf{dupl}_{L,R}$. Conversely, we show that every positive adjunction $F : \mathscr{C} \rightleftharpoons \mathscr{D} : G$ between an associative category $\mathscr{C}$ and a duploid $\mathscr{D}$ induces a $\Downarrow$-relative coadjunction (Proposition 26). (And thus dually for the construction of a relative adjunction from negative adjunction.)

Let us recall first a few fundamental and useful properties of adjunctions over non-associative categories, also developed in more detail in MMMM 2026.

Proposition 21 (R.A.P.L.)**.** *Right adjoints preserve linear maps and their composition.*

Proposition 22. *Left adjoints preserve thunkable maps and their composition.*

Proposition 23. *The left adjoint $F : \mathscr{C} \rightharpoonup \mathscr{D}$ of a positive adjunction where $\mathscr{C}$ is a category is a functor, and moreover restricts into a functor*

$$F \ : \ \mathscr{C} \longrightarrow \mathscr{P}_t$$

from the category $\mathscr{C}$ to the category $\mathscr{P}_t$ of positive thunkable maps in $\mathscr{D}$. Dually, the right adjoint $G : \mathscr{C} \rightharpoonup \mathscr{D}$ of a negative adjunction where $\mathscr{C}$ is a category is a functor, and moreover restricts into a functor

$$G \ : \ \mathscr{C} \longrightarrow \mathscr{N}_l$$

from the category $\mathscr{C}$ to the category $\mathscr{N}_l$ of negative linear maps in $\mathscr{D}$.

Constructing a positive adjunction from an R-relative coadjunction To demonstrate the construction of an adjunction on duploids from an R-relative coadjunction, we start by showing that we can construct a bare functor from the right adjoint of an R-relative coadjunction.

Proposition 24. *Let $L : \mathscr{A} \rightleftarrows \mathscr{B} : R$ be an adjunction over associative categories and $F\ {}_R\!\dashv G : \mathscr{B} \to \mathscr{C}$ be a R-relative coadjunction. Then, there exists a bare functor $G^- : \mathbf{dupl}_{L,R} \to \mathscr{C}$.*

Proof. On objects, we define G^- as G on objects of $\mathscr{B}$ and as $G \circ L$ on objects of $\mathscr{A}$. Let $f \in \mathbf{dupl}_{L,R}(X, Y)$, that is, a morphism in $\mathscr{A}(X^+, RY^-)$. The definition of $G^- f$ depends on whether X is in $\mathscr{A}$ or $\mathscr{B}$. If X is in $\mathscr{A}$, we define $G^- f$ as:

$$G^- f \overset{\text{def}}{=} G\Phi^{-1}_{X,Y^-}(f) \in \mathscr{C}(GLX, GY^-)$$

where $\Phi_{A,B} : \mathscr{B}(LA, B) \xrightarrow{\cong} \mathscr{A}(A, RB)$. If X is in $\mathscr{B}$, we define G^- as:

$$G^- f \overset{\text{def}}{=} \Psi_{GX,Y^-}(f \circ \epsilon_X) \in \mathscr{C}(GX, GY^-)$$

where $\Psi_{C,B} : \mathscr{A}(FC, RB) \xrightarrow{\cong} \mathscr{C}(C, GB)$. $\square$

Proposition 25. *Every R-relative coadjunction $F\ {}_R\!\dashv G$ of the form (9) induces a positive adjunction on $\mathbf{dupl}_{L,R}$:*

$$\begin{tikzcd} \mathscr{C} \arrow[rr, bend left, "F"] \arrow[rr, phantom, "\perp"] & & \mathbf{dupl}_{L,R} \arrow[ll, bend left, "G^-"] \end{tikzcd}$$

Proof. We construct the bijection of the adjunction $\mathbf{dupl}_{L,R}(FC, X) \cong \mathscr{C}(C, G^{-}X)$ by case analysis on X. If X is in $\mathscr{B}$, then:

$$\mathbf{dupl}_{L,R}(FC, X) = \mathscr{A}(FC, RX) \cong \mathscr{C}(C, GX) \cong \mathscr{C}(C, G^{-}X)$$

If X is in $\mathscr{A}$, then:

$$\mathbf{dupl}_{L,R}(FC, X) = \mathscr{A}(FC, RLX) \cong \mathscr{C}(C, GLX) \cong \mathscr{C}(C, G^{-}X)$$

□

Constructing a ⇓-relative coadjunction from a positive adjunction

Proposition 26. *Every positive adjunction $F : \mathscr{C} \rightleftharpoons \mathscr{D} : G$ where $\mathscr{C}$ is a category induces a ⇓-relative coadjunction of the form:*

$$\begin{tikzcd} & \mathscr{C} \arrow[dl, bend right, "F"'] \arrow[d, phantom, "\dashv"] & \\ \mathscr{P}_t & & \mathscr{N}_l \arrow[ul, bend right, "G"'] \arrow[ll, "\Downarrow"] \end{tikzcd}$$

where we write G for the restriction of the original right adjoint $G : \mathscr{D} \rightharpoonup \mathscr{C}$.

Proof. Let C be an object of $\mathscr{C}$ and N an object of $\mathscr{N}_l$. We construct the bijection of the ⇓-relative adjunction by using the bijection of the adjunction on duploids:

$$\mathscr{P}_t(FC, \Downarrow N) \cong \mathscr{D}(FC, N) \cong \mathscr{C}(C, GN)$$

□

3.3. Relative LNL coadjunctions and direct LNL adjunctions

We now consider the case where the modality is the exponential $A \mapsto\ !A$. The linear-non-linear adjunction becomes, when polarized, either a R-relative linear-non-linear coadjunction in an indirect setting or a positive adjunction with a cartesian category in a direct setting. We also show how we can construct one from the other.

Definition 27. Let $\mathscr{A}$ and $\mathscr{B}$ be two categories such that $\mathscr{A}$ is symmetric monoidal category and $R : \mathscr{B} \to \mathscr{A}$ be a functor. A ***R*-relative linear-non-linear coadjunction** is a R-relative coadjunction:

$$\begin{tikzcd} \mathscr{M} \arrow[rr, bend left, "Lin"] \arrow[rr, phantom, "\perp"] & & \mathscr{A} & & \mathscr{B} \arrow[ll, bend left, "R"] \arrow[llll, bend left=40, "Mult"] \end{tikzcd} \tag{17}$$

such that $\mathscr{M}$ is cartesian and Lin is a strong monoidal functor.

This relative coadjunction means that we have a family of bijections :

$$\varphi_{\vec{X}_i, A} : \mathscr{A}(Lin\, X_1 \otimes \cdots \otimes Lin\, X_n, RA) \cong \mathscr{M}(X_1 \times \cdots \times X_n, Mult\, A)$$

natural in $X_1, \ldots, X_n$ and A.

Definition 28. A **direct linear-non-linear adjunction** is a positive adjunction

$$\begin{tikzcd} \mathscr{M} \arrow[r, bend left, "Lin"] \arrow[r, phantom, "\perp"] & \mathscr{D} \arrow[l, bend left, "Mult"] \end{tikzcd}$$

between a symmetric monoidal duploid $\mathscr{D}$ and a cartesian category $\mathscr{M}$ such that the functor from $\mathscr{M}$ to $\mathscr{P}_t$ induced by Lin is strong monoidal.

Correspondence theorem

Proposition 29. *Every R-relative linear-non-linear coadjunction $Lin\ {}_R\!\dashv Mult$ of the form (17) induces a direct linear-non-linear adjunction:*

$$\begin{tikzcd} \mathscr{M} \arrow[rr, bend left, "Lin"] \arrow[rr, phantom, "\perp"] & & \mathbf{dupl}_{L,R} \arrow[ll, bend left, "Mult^{-}"] \end{tikzcd} \tag{18}$$

Proposition 30. *Every direct linear-non-linear adjunction $Lin : \mathscr{M} \rightleftharpoons \mathscr{D} : Mult$ induces a $\Downarrow$-relative linear-non-linear coadjunction of the form:*

$$\begin{tikzcd} & \mathscr{M} \arrow[dl, bend right, "Lin"'] \arrow[d, phantom, "\dashv"] & \\ \mathscr{P}_t & \Downarrow & \mathscr{N}_l \arrow[ul, bend right, "Mult"'] \arrow[ll] \end{tikzcd}$$

3.4. Strength for a relative monads strong with respect to a comonad

Finally, we consider the case where we have the resource modality $A \mapsto\ !A$ and an effect modality $A \mapsto \lozenge A$ strong with respect to !.

Definition 31. Suppose given an adjunction $L : \mathscr{A} \rightleftarrows \mathscr{B} : R$ where the category $\mathscr{A}$ is equipped with a symmetric monoidal structure $(\otimes, \mathbf{1})$ and where the monad $R \circ L$ is strong. Suppose also given an L-relative adjunction $F \dashv G$ as well as an R-relative linear-non-linear coadjunction $Lin \dashv Mult$. Diagramatically:

$$\begin{tikzcd} \mathscr{M} \arrow[r, bend left, "Lin"] \arrow[r, phantom, "\perp"] & \mathscr{A} \arrow[r, bend left, "L"] \arrow[r, phantom, "\perp"] \arrow[rr, bend left=40, "F"] & \mathscr{B} \arrow[l, bend left, "R"] \arrow[ll, bend left=40, "Mult"] \arrow[r, phantom, "\perp"] & \mathscr{C} \arrow[l, bend left, "G"] \end{tikzcd} \tag{19}$$

The relative monad $\lozenge = G \circ F$ is strong relatively to the functor Lin if it is equipped with a natural family of morphisms in $\mathscr{A}$

$$str_{X,A} : Lin\, X \otimes R\lozenge A \to R\lozenge(Lin\, X \otimes A)$$

making the four usual coherence diagrams commute.

Definition 32. Let a direct linear-non-linear adjunction $Lin : \mathscr{M} \rightleftharpoons \mathscr{D} : Mult$ and a negative adjunction $F : \mathscr{D} \rightleftharpoons \mathscr{C} : G$. Diagrammatically:

$$\begin{tikzcd} \mathscr{M} \arrow[r, bend left, "Lin"] \arrow[r, phantom, "\perp"] & \mathscr{D} \arrow[l, bend left, "Mult"] \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp"] & \mathscr{C} \arrow[l, bend left, "G"] \end{tikzcd} \tag{20}$$

The bare endofunctor $\Diamond = G \circ F$ is strong relatively to the bare functor Lin if it is equipped with a natural family of morphisms in $\mathscr{D}$

$$str_{X,A} : Lin\,X \otimes \Diamond A \to \Diamond(Lin\,X \otimes A)$$

linear wrt. $\Diamond A$ (MMMM 2026, Definition F.2, p.42), making the four usual coherence diagrams commute.

Correspondence theorem

Proposition 33. *Every situation of the form* (19) *such that the L-relative monad $G \circ F$ is strong relatively to Lin induces a negative adjunction:*

$$\begin{tikzcd} \mathbf{dupl}_{L,R} \arrow[r, bend left, "F^+"] \arrow[r, phantom, "\perp"] & \mathscr{C} \arrow[l, bend left, "G"] \end{tikzcd}$$

such that $G \circ F^+$ is strong relatively to Lin, the left adjoint of the induced direct linear-non-linear adjunction from (18)*.*

Proposition 34. *Every situation of the form* (20) *such that the bare endofunctor $G \circ F$ is strong relatively to Lin induces a $\Uparrow$-relative adjunction:*

$$\begin{tikzcd} & \mathscr{C} \arrow[dr, bend left, "G"] \arrow[d, phantom, "\dashv"] & \\ \mathscr{P}_t \arrow[ur, bend left, "F"] \arrow[rr, "\Uparrow"'] & {} & \mathscr{N}_l \end{tikzcd}$$

such that the relative monad $G \circ F$ is strong relatively to Lin.

Corollary 35. *Any* $\mathbf{ILL}_p^{\Diamond}$ *model over an adjunction $\uparrow : \mathscr{V} \rightleftarrows \mathscr{S} : \downarrow$ induces an* $\mathbf{ILL}_p^{\Diamond}$ *model over its completion $\Uparrow : \mathscr{P}_t \rightleftarrows \mathscr{N}_l : \Downarrow$.*

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Abbreviations

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A. Appendix

A.1. Componentwise bare and proper functors

A componentwise bare functor

$$F \quad : \quad \mathscr{A}, \mathscr{B} \longrightarrow \mathscr{C}$$

between non associative categories $\mathscr{A}$, $\mathscr{B}$ and $\mathscr{C}$, is a pair consisting of a function

$$F_{\mathsf{obj}} \quad : \quad \mathscr{A}_{\mathsf{obj}} \times \mathscr{B}_{\mathsf{obj}} \longrightarrow \mathscr{C}_{\mathsf{obj}}$$

which associates an object $F(A, B)$ of $\mathscr{C}$ to every pair (A, B) consisting of an object A of $\mathscr{A}$ and of an object B of $\mathscr{B}$; together with a family of functions

$$F^{B}_{\mathsf{fst}}(A_1, A_2) \quad : \quad \mathscr{A}(A_1, A_2) \longrightarrow \mathscr{C}((A_1, B), (A_2, B))$$

$$F^{A}_{\mathsf{snd}}(B_1, B_2) \quad : \quad \mathscr{B}(B_1, B_2) \longrightarrow \mathscr{C}((A, B_1), (A, B_2))$$

such that the series of equalities holds:

$$F^{B}_{\mathsf{fst}}(\mathrm{id}A) = \mathrm{id}_{F_{\mathsf{obj}}(A,B)} \qquad F^{A}_{\mathsf{snd}}(\mathrm{id}B) = \mathrm{id}_{F_{\mathsf{obj}}(A,B)}$$

A componentwise proper functor is a componentwise bare functor such that the equality holds:

$$F^{B}_{\mathsf{fst}}(g \circ f) = F^{B}_{\mathsf{fst}}(g) \circ F^{B}_{\mathsf{fst}}(f)$$

$$F^{A}_{\mathsf{snd}}(g \circ f) = F^{A}_{\mathsf{snd}}(g) \circ F^{A}_{\mathsf{snd}}(f)$$

A.2. Tensor product

The tensor product $\mathscr{A} \boxtimes \mathscr{B}$ of two non associative categories $\mathscr{A}$ and $\mathscr{B}$ is the non associative category whose objects are pairs (A, B)

$$(A, g) \quad : \quad (A, B_1) \longrightarrow (A, B_2)$$

$$(f, B) \quad : \quad (A_1, B) \longrightarrow (A_2, B)$$

such that

$$(A, g_2 \circ g_1) = (A, g_2) \circ (A, g_1) \qquad (A, \mathrm{id}B) = \mathrm{id}(A, B)$$
$$(f_2 \circ f_1, B) = (f_2, B) \circ (f_1, B) \qquad (\mathrm{id}A, B) = \mathrm{id}(A, B)$$

There is a componentwise proper functor

$$\mathscr{A}, \mathscr{B} \longrightarrow \mathscr{A} \boxtimes \mathscr{B}$$

which induces a bijection between componentwise proper functors

$$\mathscr{A}, \mathscr{B} \longrightarrow \mathscr{C}$$

and proper functors

$$\mathscr{A} \boxtimes \mathscr{B} \longrightarrow \mathscr{C}$$

Moreover, it induces a bijection between componentwise bare functors

$$\mathscr{A}, \mathscr{B} \longrightarrow \mathscr{C}$$

and bare functors

$$\mathscr{A} \boxtimes \mathscr{B} \longrightarrow \mathscr{C}$$

B. $\mathbf{IMALL}_p$ has sums

Definition 36. A non-associative category $\mathscr{D}$ has (binary) sums if Δ has a positive left adjoint, in other words a bare functor $\oplus : \mathscr{D} \times \mathscr{D} \rightharpoonup \mathscr{D}$ such that $A \oplus B$ is positive and such that there exists a family of isomorphisms:

$$\mathscr{D}(A \oplus B, C) \cong \mathscr{D}(A, C) \times \mathscr{D}(B, C)$$

which is natural in $(\mathscr{D} \times \mathscr{D})^{\mathrm{op}} \boxtimes \mathscr{D} \rightharpoonup \mathcal{S}et$.

Let us spell out the naturality condition more explicitly: for any $f : A \to C$ and $g : B \to C$ there exists a pairing $[f, g] : A \oplus B \to C$ such that:

$$\forall h : C \to D, \qquad h \circledcirc [f, g] = [h \circledcirc f, h \circledcirc g] \tag{21}$$

$$\forall h_1 : A' \to A, \forall h_2 : B' \to B, \qquad [f, g] \circledcirc (h_1 \oplus h_2) = [f \circledcirc h_1, g \circledcirc h_2] \tag{22}$$

Proposition 37. *The syntactic duploid has sums given by the type $\oplus$ and its constructors:*

$$\begin{aligned}
\oplus_{A_x, B_y, C, D, z} &: \mathscr{D}(A_x, C) \times \mathscr{D}(B_y, D) \to \mathscr{D}((A \oplus B)_z, C \oplus D) \\
t \oplus_{x,y,z} u &\stackrel{\text{def}}{=} \mu\alpha.\langle z \,\|\, \tilde{\mu}[x.\langle \iota_1(t) \,\|\, \alpha\rangle \,|\, y.\langle \iota_2(u) \,\|\, \alpha\rangle]\rangle \\
&\quad \textit{where } \iota_i(s) \stackrel{\text{def}}{=} \mu\alpha.\langle t \,\|\, \tilde{\mu}x.\langle \iota_i(x) \,\|\, \alpha\rangle\rangle \\
\varphi_{A,B,C} &: \begin{cases} \mathscr{D}(A \oplus B, C_\alpha) \to \mathscr{D}(A, C_\alpha) \times \mathscr{D}(B, C_\alpha) \\ e \mapsto (\tilde{\mu}x.\langle \iota_1(x) \,\|\, e\rangle, \tilde{\mu}y.\langle \iota_2(y) \,\|\, e\rangle) \end{cases} \\
\psi_{A,B,C} &: \begin{cases} \mathscr{D}(A, C_\alpha) \times \mathscr{D}(B, C_\alpha) \to \mathscr{D}(A \oplus B, C_\alpha) \\ (e_1, e_2) \mapsto [e_1, e_2] \stackrel{\text{def}}{=} \tilde{\mu}[x.\langle x \,\|\, e_1\rangle \,|\, y.\langle y \,\|\, e_2\rangle] \end{cases}
\end{aligned}$$

Proof.

- Positivity: (by definition, and in fact) from the observation that expansion at type $A \oplus B$ implies the linearity of all $e \in \mathscr{D}(A \oplus B, C)$.

- $(\psi \circ \varphi)_{A,B,C} = \mathrm{id}_{\mathscr{D}(A \oplus B, C)}$ amounts to the equality between terms:

$$\begin{aligned}
&\tilde{\mu}\big[x.\langle x \,\|\, \tilde{\mu}x.\langle \iota_1(x) \,\|\, e\rangle\rangle \,\big|\, y.\langle y \,\|\, \tilde{\mu}y.\langle \iota_2(y) \,\|\, e\rangle\rangle\big] \\
&\simeq_{\mathrm{R}} \tilde{\mu}[x.\langle \iota_1(x) \,\|\, e\rangle \,|\, y.\langle \iota_2(y) \,\|\, e\rangle] \\
&\simeq_{\mathrm{E}} e
\end{aligned}$$

- $(\varphi \circ \psi)_{A,B,C} = \mathrm{id}_{\mathscr{D}(A,C) \times \mathscr{D}(B,C)}$ amounts to the equality between terms:

$$\begin{aligned}
&\tilde{\mu}y.\langle \iota_i(y) \,\|\, \tilde{\mu}[x_1.\langle x_1 \,\|\, e_1\rangle \,|\, x_2.\langle x_2 \,\|\, e_2\rangle]\rangle \\
&\simeq_{\mathrm{R}} \tilde{\mu}y.\langle y \,\|\, e_i\rangle \\
&\simeq_{\mathrm{e}} e_i
\end{aligned}$$

- The naturality condition (21) amounts, for all $e_1 \in \mathscr{D}(A, C_\alpha)$, $e_2 \in \mathscr{D}(B, C_\alpha)$ and $e_3 \in \mathscr{D}(C, D)$, to the equality between terms:

$$\begin{aligned}
c_1 &\stackrel{\text{def}}{=} \big\langle \mu\alpha.\langle x \,\|\, \tilde{\mu}[y.\langle y \,\|\, e_1\rangle \,|\, z.\langle z \,\|\, e_2\rangle]\rangle \,\big\|\, e_3\big\rangle \\
c_2 &\stackrel{\text{def}}{=} \big\langle x \,\big\|\, \tilde{\mu}\big[y.\langle \mu\alpha.\langle y \,\|\, e_1\rangle \,\|\, e_3\rangle \,\big|\, z.\langle \mu\alpha.\langle z \,\|\, e_2\rangle \,\|\, e_3\rangle\big]\big\rangle \\
c_1 &\simeq_{\mathrm{RE}} c_2
\end{aligned}$$

when e_3 is not necessarily linear. It holds thanks to $\rhd_{\mathrm{E}}$:

$$\begin{aligned}
\bar{\mu}x.c_1 &\rhd_{\mathrm{E}} \bar{\mu}\big[y.\langle\iota_1(y)\,\|\,\bar{\mu}x.c_1\rangle\,\big|\,z.\langle\iota_2(z)\,\|\,\bar{\mu}x.c_1\rangle\big]\\
&\to_{\mathrm{R}}^{*} \bar{\mu}\Big[y.\big\langle\mu\alpha.\langle y\,\|\,e_1\rangle\,\|\,e_3\big\rangle\,\Big|\,z.\big\langle\mu\alpha.\langle z\,\|\,e_2\rangle\,\|\,e_3\big\rangle\Big]\\
&\lhd_{\mathrm{r}} \bar{\mu}x.\Big\langle x\,\Big\|\,\bar{\mu}\Big[y.\big\langle\mu\alpha.\langle y\,\|\,e_1\rangle\,\|\,e_3\big\rangle\,\Big|\,z.\big\langle\mu\alpha.\langle z\,\|\,e_2\rangle\,\|\,e_3\big\rangle\Big]\Big\rangle\\
&= \bar{\mu}x.c_2
\end{aligned}$$

- The naturality condition (22) amounts, for all $e_1 \in \mathscr{D}(A, C_\alpha)$, $e_2 \in \mathscr{D}(B, C_\alpha)$, $t \in \mathscr{D}(A'_x, A)$, $u \in \mathscr{D}(B'_y, B)$, to the equality between terms:

$$\begin{aligned}
c_1 &\stackrel{\text{def}}{=} \langle t \oplus_{x,y,z} u\,\|\,[e_1, e_2]\rangle\\
c_2 &\stackrel{\text{def}}{=} \big\langle z\,\big\|\,\bar{\mu}\big[x.\langle t\,\|\,e_1\rangle\,\big|\,y.\langle u\,\|\,e_2\rangle\big]\big\rangle\\
c_1 &\simeq_{\mathrm{RE}} c_2
\end{aligned}$$

It holds indeed:

$$\begin{aligned}
\langle t \oplus_{x,y,z} u\,\|\,[e_1, e_2]\rangle &= \Big\langle\mu\alpha.\big\langle z\,\big\|\,\bar{\mu}\big[x.\langle\iota_1(t)\,\|\,\alpha\rangle\,\big|\,y.\langle\iota_2(u)\,\|\,\alpha\rangle\big]\big\rangle\,\Big\|\,[e_1, e_2]\Big\rangle\\
&\rhd_{\mathrm{r}} \big\langle z\,\big\|\,\bar{\mu}\big[x.\langle\iota_1(t)\,\|\,[e_1, e_2]\rangle\,\big|\,y.\langle\iota_2(u)\,\|\,[e_1, e_2]\rangle\big]\big\rangle\\
&\to^{*} \big\langle z\,\big\|\,\bar{\mu}\big[x.\langle t\,\|\,e_1\rangle\,\big|\,y.\langle u\,\|\,e_2\rangle\big]\big\rangle\\
&= c_2
\end{aligned}$$

□

C. Relative monads and comonads in $\mathbf{ILL}_p^{\Diamond}$

Definition 38 (Altenkirch, Chapman and Uustalu, 2015)**.** A relative monad T on a functor $L : \mathscr{A} \to \mathscr{B}$ is given by

- a mapping on objects $T : |\mathscr{A}| \to |\mathscr{B}|$,
- for any $A \in \mathscr{A}$, a map $\eta_A \in \mathscr{B}(LA, TA)$,
- for any $A, B \in \mathscr{A}$, and any $f \in \mathscr{B}(LA, TB)$, a map $f^* \in \mathscr{B}(TA, TB)$,

such that two unital laws and one associativity law are satisfied:

$$f = f^* \circ \eta \qquad\qquad \eta_A^* = \mathrm{id}_{TA} \qquad\qquad (f^* \circ g)^* = f^* \circ g^*$$

By duality:

Definition 39. A relative comonad S on a functor $R : \mathscr{B} \to \mathscr{A}$ is given by:

- a mapping on objects $S : |\mathscr{B}| \to |\mathscr{A}|$,
- for any $B \in \mathscr{B}$, a map $\varepsilon_A \in \mathscr{A}(SA, RA)$,
- for any $A, B \in \mathscr{B}$, and any $f \in \mathscr{A}(SA, RB)$, a map ${}^*f \in \mathscr{A}(SA, SB)$,

such that two unital laws and one associativity law are satisfied:

$$f = \varepsilon \circ {}^*f \qquad\qquad {}^*\varepsilon_B = \mathrm{id}_{SB} \qquad\qquad {}^*(f \circ {}^*g) = {}^*f \circ {}^*g$$

We now prove that the syntactic linear CBPV model $\mathscr{V} \rightleftarrows \mathscr{S}$ of thunkable and linear terms of $\mathbf{ILL}_p^{\Diamond}$ has a relative comonad $!$ on $\downarrow : \mathscr{S} \to \mathscr{V}$.

A thunkable map $!N \vdash \downarrow M$ of the syntactic model is the same thing as a term with judgement $!N \vdash M$ in $\mathbf{ILL}_p^{\Diamond}$. The co-unit ε_N is thus given by the following covalue:

$$\mid\ !\alpha : !N \vdash \alpha : N$$

Given any thunkable map $!N \vdash \downarrow M$, in the form of a command

$$c : (x : !N \vdash \alpha : M)\,,$$

its coextension ${}^*c$ is the value:

$$x : !N \vdash \underbrace{\mu!\alpha.c}_{{}^*c} : !M$$

which is indeed thunkable by definition. The two unital laws

$$\varepsilon \circ {}^*f = f \qquad\qquad \mathrm{id}_{SB} = {}^*\varepsilon_B$$

coincide respectively with the following instances of reduction and expansion:

$$\langle \mu!\alpha.c \,\|\, !\alpha \rangle^+ \rhd_{\mathrm{R}} c \qquad\qquad x \rhd_{\mathrm{E}} \mu!\alpha.\langle x \,\|\, !\alpha \rangle$$

The associativity law:

$${}^*(f \circ {}^*g) = {}^*f \circ {}^*g$$

instantiated for $c : (x : !N_1 \vdash \alpha : N_2)$ and $c' : (y : !N_2 \vdash \beta : N_3)$ corresponds to

$$\mu!\beta.\langle \mu!\alpha.c \,\|\, \tilde{\mu}y^+.c' \rangle^+ \simeq_{\mathrm{RE}} \mu\gamma^+.\langle \mu!\alpha.c \,\|\, \tilde{\mu}y^+.\langle \mu!\beta.c' \,\|\, \gamma \rangle^+ \rangle^+$$

which indeed holds given that both sides are convertible to $\mu!\beta.c'[\mu!\alpha.c/y]$.

The case of the relative monad $\Diamond : \mathscr{V} \to \mathscr{S}$ on $\uparrow : \mathscr{V} \to \mathscr{S}$ is symmetric. This establishes Proposition 16.

Proposition 40. *A term $x : !A \vdash t : !B$ is a $!$-coalgebra morphism in the syntactic model if and only if the following commutation holds for any c:*

$$\langle t \,\|\, \tilde{\mu}y^+.\langle \mu!\beta.c \,\|\, \alpha \rangle^+ \rangle^+ \simeq_{\mathrm{RE}} \langle \mu!\beta.\langle t \,\|\, \tilde{\mu}y^+.c \rangle^+ \,\|\, \alpha \rangle^+$$

Dually, a term $e : \Diamond A \vdash \alpha : \Diamond B$ is a $\Diamond$-algebra morphism in the syntactic model if and only if the following commutation holds for any c:

$$\langle \mu\beta^-.\langle x \,\|\, \tilde{\mu}\Diamond y.c \rangle^- \,\|\, e \rangle^- \simeq_{\mathrm{RE}} \langle x \,\|\, \tilde{\mu}\Diamond y.\langle \mu\beta^-.c \,\|\, e \rangle^- \rangle^-\,.$$

These characterisations extend beyond free (co)algebras if we add other types of (co)algebras to the type system making these commutations well typed.