

1 Focusing

$$\begin{aligned}
& (\Gamma \vdash A \rightarrow B) > \{(\Gamma, A \vdash B)\} \\
& (\Gamma, A \otimes B, \Gamma' \vdash \Delta) > \{(\Gamma, A, B, \Gamma' \vdash \Delta)\} \\
& (\Gamma, \mathbf{1}, \Gamma' \vdash \Delta) > \{(\Gamma, \Gamma' \vdash \Delta)\} \\
& (\Gamma \vdash A \& B) > \{(\Gamma \vdash A), (\Gamma \vdash B)\} \\
& (\Gamma, A \oplus B, \Gamma' \vdash \Delta) > \{(\Gamma, A, \Gamma' \vdash \Delta), (\Gamma, B, \Gamma' \vdash \Delta)\} \\
& (\Gamma \vdash \top) > \emptyset \\
& (\Gamma, \mathbf{0}, \Gamma' \vdash \Delta) > \emptyset \\
& \mathcal{M} \uplus \{\Phi\} >_{\text{m}} \mathcal{M} \uplus \mathcal{N} \text{ whenever } \Phi > \mathcal{N}
\end{aligned}$$

Figure 1: Inversion relation between multisets of sequents

$$\begin{aligned}
c : (\Gamma, z : A \otimes B, \Gamma' \vdash \Delta) & > \{\langle x \otimes y \parallel \bar{\mu}z.c \rangle : (\Gamma, x : A, y : B, \Gamma' \vdash \Delta)\} \\
c : (\Gamma, z : A \oplus B, \Gamma' \vdash \Delta) & > \{\langle \iota_1(x) \parallel \bar{\mu}z.c \rangle : (\Gamma, x : A, \Gamma' \vdash \Delta), \langle \iota_2(y) \parallel \bar{\mu}z.c \rangle : (\Gamma, y : B, \Gamma' \vdash \Delta)\} \\
c : (\Gamma, x : \mathbf{1}, \Gamma' \vdash \Delta) & > \{\langle () \parallel \bar{\mu}x.c \rangle : (\Gamma, \Gamma' \vdash \Delta)\} \\
c : (\Gamma \vdash \beta : A \rightarrow B) & > \{\langle \mu\beta.c \parallel x.\alpha \rangle : (\Gamma, x : A \vdash \alpha : B)\} \\
c : (\Gamma \vdash \gamma : A \& B) & > \{\langle \mu\gamma.c \parallel \pi_1.\alpha \rangle : (\Gamma \vdash \alpha : A), \langle \mu\gamma.c \parallel \pi_2.\beta \rangle : (\Gamma \vdash \beta : B)\} \\
c : (\Gamma, x : \mathbf{0}, \Gamma' \vdash \Delta) & > \emptyset \\
c : (\Gamma \vdash \alpha : \top) & > \emptyset
\end{aligned}$$

Figure 2: Inversion step of a command

$$\begin{array}{c}
\frac{c : (\Gamma, x : A, y : B, \Gamma' \vdash \Delta)}{\langle z \parallel \bar{\mu}(x \otimes y).c \rangle : (\Gamma, z : A \otimes B, \Gamma' \vdash \Delta)} \qquad \frac{c : (\Gamma, x : A \vdash \alpha : B)}{\langle \mu(x \cdot \alpha).c \parallel \beta \rangle : (\Gamma \vdash \beta : A \rightarrow B)} \\
\frac{c : (\Gamma, x : A, \Gamma' \vdash \Delta) \quad c' : (\Gamma, y : B, \Gamma' \vdash \Delta)}{\langle z \parallel \bar{\mu}[x.c \mid y.c'] \rangle : (\Gamma, z : A \oplus B, \Gamma' \vdash \Delta)} \qquad \frac{c : (\Gamma \vdash \alpha : A) \quad c' : (\Gamma \vdash \beta : B)}{\langle \mu \langle \alpha.c ; \beta.c' \rangle \parallel \gamma \rangle : (\Gamma \vdash \gamma : A \& B)} \\
\frac{c : (\Gamma, \Gamma' \vdash \Delta)}{\langle x \parallel \bar{\mu}().c \rangle : (\Gamma, x : \mathbf{1}, \Gamma' \vdash \Delta)} \\
\frac{}{\langle x \parallel \bar{\mu}[\]_{\Gamma, \Gamma', \Delta} \rangle : (\Gamma, x : \mathbf{0}, \Gamma' \vdash \Delta)} \qquad \frac{}{\langle \mu \langle \rangle_{\Gamma} \parallel \alpha \rangle : (\Gamma \vdash \alpha : \top)}
\end{array}$$

Figure 3: Inversion phase

$$\begin{array}{c}
\frac{c : (\Gamma \vdash \alpha : N)}{\Gamma \vdash \mu \alpha^{\ominus}.c : N} \qquad \frac{c : (\Gamma, x : P, \Gamma' \vdash \Delta)}{\Gamma, \Gamma' \mid \bar{\mu} x^{\oplus}.c : P \vdash \Delta} \\
\frac{}{\Gamma, x : X^{\oplus}, \Gamma' \vdash x : X^{\oplus}} \qquad \frac{}{\Gamma \mid \alpha : X^{\ominus} \vdash \alpha : X^{\ominus}} \\
\frac{\Gamma \vdash V : A \mid \Gamma \vdash W : B}{\Gamma \vdash V \otimes W : A \otimes B} \qquad \frac{\Gamma \vdash V : A \mid \Gamma \mid S : B \vdash \Delta}{\Gamma \mid V \cdot S : A \rightarrow B \vdash \Delta} \\
\frac{}{\Gamma \vdash () : \mathbf{1}} \qquad \frac{\Gamma \vdash V : A_i}{\Gamma \vdash i_i(V) : A_1 \oplus A_2} \qquad \frac{\Gamma \mid S : A_i \vdash \Delta}{\Gamma \mid \pi_i \cdot S : A_1 \& A_2 \vdash \Delta} \\
\frac{\Gamma \vdash V : P}{\langle V \parallel \alpha \rangle^{\oplus} : (\Gamma \vdash \alpha : P)} \qquad \frac{\Gamma, x : N \mid S : N \vdash \Delta}{\langle x \parallel S \rangle^{\ominus} : (\Gamma, x : N \vdash \Delta)}
\end{array}$$

Figure 4: Focusing phase in \mathbf{LJ}_p^{η}

2 LL

Formulae: $A, B, A_\epsilon ::= P \mid N$
 Positives: $P, Q, A_+ ::= X \mid \mathbf{1} \mid A \otimes B \mid A \oplus B \mid \mathbf{0}$
 Negatives: $N, M, A_- ::= \bar{X} \mid \perp \mid A \wp B \mid A \& B \mid \top$

(a) Formulae

$$\begin{array}{ccccc} \overline{A \otimes B} \stackrel{\text{def}}{=} \bar{A} \wp \bar{B} & \bar{\mathbf{1}} \stackrel{\text{def}}{=} \perp & \overline{A \oplus B} \stackrel{\text{def}}{=} \bar{A} \& \bar{B} & \bar{\mathbf{0}} \stackrel{\text{def}}{=} \top & \overline{(X)} \stackrel{\text{def}}{=} \bar{X} \\ \overline{A \wp B} \stackrel{\text{def}}{=} \bar{A} \otimes \bar{B} & \bar{\perp} \stackrel{\text{def}}{=} \mathbf{1} & \overline{A \& B} \stackrel{\text{def}}{=} \bar{A} \oplus \bar{B} & \bar{\top} \stackrel{\text{def}}{=} \mathbf{0} & \overline{\bar{X}} \stackrel{\text{def}}{=} X \end{array}$$

(b) Dual of a formula

Figure 5: **MALL** formulae and duality

(Co)Values: $V, W ::= x, y, \dots \mid \mu x^\ominus.c \mid \begin{array}{c} \mathbf{1}/\perp \\ \mu().c \end{array} \mid \begin{array}{c} \otimes/\wp \\ \mu(x \wp y).c \end{array} \mid \begin{array}{c} \oplus/\& (i \in \{1,2\}) \\ \iota_i(V) \end{array} \mid \begin{array}{c} \mathbf{0}/\top \\ \mu[V] \end{array}$

(Co)Expressions: $t, u ::= V \mid \mu x^+.c$

Commands: $c ::= \langle t \parallel V \rangle \quad (\stackrel{\text{not.}}{=} \langle V \parallel t \rangle)$

(a) Terms

$$\begin{array}{lll} (R\mu^\epsilon) : & \langle \mu x^+.c \parallel V \rangle & \triangleright_R \quad c[V/x] \\ (R\mu^\ominus) : & \langle V \parallel \mu x^\ominus.c \rangle & \triangleright_R \quad c[V/x] \\ (R\mathbf{1}/\perp) : & \langle () \parallel \mu().c \rangle & \triangleright_R \quad c \\ (R\otimes/\wp) : & \langle V \otimes W \parallel \mu(x \wp y).c \rangle & \triangleright_R \quad c[V/x, W/y] \\ (R\oplus/\&) : & \langle \iota_i(V) \parallel \mu[x_1.c_1 \mid x_2.c_2] \rangle & \triangleright_R \quad c_i[V/x_i] \end{array}$$

(no rule **R0**/ \top)

(b) Reduction rules

$$\begin{array}{lll} (E\mu^+) : & \mu x^+.\langle t \parallel x \rangle & \triangleright_E \quad t \\ (E\mu^\ominus) : & \mu x^\ominus.\langle x \parallel V \rangle & \triangleright_E \quad V \\ (E\mathbf{1}/\perp) : & \mu().\langle () \parallel V \rangle & \triangleright_E \quad V \\ (E\otimes/\wp) : & \mu(x \wp y).\langle x \otimes y \parallel V \rangle & \triangleright_E \quad V \\ (E\oplus/\&) : & \mu[x.\langle \iota_1(x) \parallel V \rangle \mid y.\langle \iota_2(y) \parallel V \rangle] & \triangleright_E \quad V \\ (E\mathbf{0}/\top)^\ddagger : & \mu[x_1 \otimes \dots \otimes x_n] & \triangleright_E \quad V \end{array}$$

(c) Extensionality rules (\ddagger : the η rule for $\mathbf{0}/\top$ is meaningless without a valid typing judgement)

Figure 6: **MALL** $^\eta$: calculus

- $\Gamma, \Delta \dots$ are maps from a finite set of variables to types provided with total orders on their domain, notation $(x_1 : A_1, \dots, x_n : A_n)$.
- Concatenation (Γ, Δ) is defined when the domains of Γ and Δ are disjoint.
- $\Sigma^*(\Gamma; \Gamma')$ is the set of bijections $\sigma : \text{dom } \Gamma \rightarrow \text{dom } \Gamma'$ satisfying $\Gamma'(\sigma(x)) = \Gamma(x)$ for all $x \in \text{dom } \Gamma$.
- Judgements are: $c : (\vdash \Gamma) \quad \vdash t : A \mid \Gamma$
(a) Judgements

$$\begin{array}{c}
\frac{}{\vdash x : A \mid x : \bar{A}} \text{(ax)} \quad \frac{c : (\vdash x : A_\varepsilon, \Gamma)}{\vdash \mu x^\varepsilon . c : A_\varepsilon \mid \Gamma} (\mu^\varepsilon) \quad \frac{\vdash t : A \mid \Gamma}{\vdash t[\sigma] : A \mid \Gamma'} (\vdash \sigma) \\
\\
\frac{\vdash t : A \mid \Gamma \quad \vdash u : \bar{A} \mid \Delta}{\langle t \parallel u \rangle : (\vdash \Gamma, \Delta)} \text{(cut)} \quad \frac{c : (\vdash \Gamma)}{c[\sigma] : (\vdash \Gamma')} (\sigma) \\
\text{(b) Identity} \quad \text{(c) Structure — } \sigma \in \Sigma^*(\Gamma; \Gamma') \\
\\
\frac{\vdash V : A \mid \Gamma \quad \vdash W : B \mid \Delta}{\vdash V \otimes W : A \otimes B \mid \Gamma, \Delta} (\otimes^f) \quad \frac{}{\vdash () : \mathbf{1} \mid} (\mathbf{1}) \\
\\
\frac{c : (\vdash x : A, y : B, \Gamma)}{\vdash \mu(x \wp y) . c : A \wp B \mid \Gamma} (\wp) \quad \frac{c : (\vdash \Gamma)}{\Gamma \vdash \mu().c : \perp \mid \Delta} (\perp) \\
\\
\frac{c : (\vdash x : A, \Gamma) \quad c' : (\vdash y : B, \Gamma)}{\vdash \mu[x.c \mid y.c'] : A \& B \mid \Gamma} (\&) \quad \frac{\Gamma \vdash V : A \mid}{\Gamma \vdash \mu[V] : \top \mid} (\top^f) \\
\\
\frac{\vdash V : A_i \mid \Gamma}{\vdash l_i(V) : A_1 \oplus A_2 \mid \Gamma} (\oplus_i^f) \quad \text{(no rule for } \mathbf{0}) \\
\text{(d) Logic}
\end{array}$$

Figure 7: MALL_p^n : simple types

$$\begin{array}{c}
\frac{\vdash t : A \mid \Gamma \quad \vdash u : B \mid \Delta}{\vdash t \otimes u : A \otimes B \mid \Gamma, \Delta} (\vdash \otimes) \quad t_\varepsilon \otimes u_{\varepsilon'} \stackrel{\text{def}}{=} \mu x^+ . \langle t \parallel \mu y^\varepsilon . \langle u \parallel \mu z^{\varepsilon'} . \langle y \otimes z \parallel x \rangle \rangle \rangle \\
\\
\frac{\Gamma \vdash t : (A_i)_\varepsilon \mid}{\Gamma \vdash l_i(t) : A_1 \oplus A_2 \mid} (\vdash \oplus_i) \quad l_i(t_\varepsilon) \stackrel{\text{def}}{=} \mu x^+ . \langle t \parallel \mu y^\varepsilon . \langle l_i(y) \parallel x \rangle \rangle \\
\\
\frac{}{\Gamma \vdash \mu[]_\Gamma : \top \mid} (\vdash \top) \quad \mu[]_\Gamma \stackrel{\text{def}}{=} \mu[x_1 \otimes \dots \otimes x_n] \text{ pour } \{x_1, \dots, x_n\} = \text{dom } \Gamma
\end{array}$$

Figure 8: Remaining rules of MALL (without value restriction)

Positives: $P, Q, A_+ ::= \dots \mid !A$ $\overline{!A} \stackrel{\text{def}}{=} \overline{?A} \quad \overline{?A} \stackrel{\text{def}}{=} \overline{!A}$
Negatives: $N, M, A_- ::= \dots \mid ?A$ (b) Dual of a formula
(a) Formulae

(Co)Values: $V, W ::= \dots \mid ?V \mid \mu!x.c$
(c) Terms

$(R!/?) : \langle \mu!x.c \parallel ?V \rangle \triangleright_R c[V/x]$ $(E!/?) : \mu!x.\langle V \parallel ?x \rangle \triangleright_E V$
(d) Reduction rules (e) Extensionality rules

$$\frac{\vdash t : A \mid \Gamma}{\vdash t[\sigma] : A \mid \Gamma'} (\vdash \sigma) \qquad \frac{c : (\vdash \Gamma)}{c[\sigma] : (\vdash \Gamma')} (\sigma)$$

(f) Structure for $\sigma \in \Sigma^!(\Gamma; \Gamma')$: the subset of $\Sigma(\Gamma; \Gamma')$ of maps which are bijective on variables not of the form $!A$.

$$\frac{c : (\vdash x : A, ?\Gamma)}{\vdash \mu!x.c : !A \mid ?\Gamma} (\text{prom.}) \qquad \frac{\vdash V : A \mid \Gamma}{\vdash ?V : ?A \mid \Gamma} (\text{der.}^f)$$

(g) Logic — where $!(x_1 : A_1, \dots, x_n : A_n)$ stands for the typing context $(x_1 : !A_1, \dots, x_n : !A_n)$.

$$\frac{\vdash t : A \mid \Gamma}{\vdash ?t : ?A \mid \Gamma} (\text{der.}) \qquad ?t_\varepsilon \stackrel{\text{def}}{=} \mu x^\ominus . \langle \mu y^\varepsilon . \langle x \parallel ?y \rangle \parallel t \rangle$$

(h) Remaining rule of **LL** (without value restriction)

Figure 9: $\mathbf{LL}_p^\eta = \mathbf{MALL}_p^\eta$ + the above