1 MALL (reminder) and LK

$$\begin{array}{lrcl} \text{Formulae:} & A, B, A_\varepsilon & ::= & P \mid N \\ \text{Positives:} & P, Q, A_+ & ::= & X \mid \mathbf{1} \mid \mathbf{0} \mid A \otimes B \mid A \oplus B \\ \text{Negatives:} & N, M, A_- & ::= & \overline{X} \mid \bot \mid \top \mid A \parr B \mid A \,\&\, B \end{array}$$

(a) Formulae

$$\begin{array}{lllll} \overline{A \otimes B} \stackrel{\text{def}}{=} \overline{A} \parr \overline{B} & \overline{\mathbf{1}} \stackrel{\text{def}}{=} \bot & \overline{A \oplus B} \stackrel{\text{def}}{=} \overline{A} \,\&\, \overline{B} & \overline{\mathbf{0}} \stackrel{\text{def}}{=} \top & \overline{(X)} \stackrel{\text{def}}{=} \overline{X} \\ \overline{A \parr B} \stackrel{\text{def}}{=} \overline{A} \otimes \overline{B} & \overline{\bot} \stackrel{\text{def}}{=} \mathbf{1} & \overline{A \,\&\, B} \stackrel{\text{def}}{=} \overline{A} \oplus \overline{B} & \overline{\top} \stackrel{\text{def}}{=} \mathbf{0} & \overline{\overline{X}} \stackrel{\text{def}}{=} X \end{array}$$

(b) Dual of a formula

Figure 1: $\mathbf{MALL}/\mathbf{LK}^\eta_p$ formulae and duality

$$\begin{array}{lrcllllll} & & & & & \mathbf{1}/\bot & \otimes/\parr & \oplus/\& \,(i\in\{1,2\}) & \mathbf{0}/\top \\ \text{(Co)Values:} & V, W & ::= & x, y, \ldots & \mid \mu x^\ominus.c & \mid () & \mid V \otimes W & \mid \iota_i(V) & \mid \mu[V] \\ & & & & & \mid \mu().c & \mid \mu(x \parr y).c & \mid \mu[x.c \mid y.c'] & \\ \text{(Co)Expressions:} & t, u & ::= & V & \mid \mu x^+.c & & & & \\ \text{Commands:} & c & ::= & \langle t \parallel V \rangle & (\stackrel{\text{not.}}{=} \langle V \parallel t \rangle) & & & & \end{array}$$

(a) Terms

$$\begin{array}{rccl} (R\mu^\varepsilon): & \langle \mu x^+.c \parallel V \rangle & \rhd_{\text{R}} & c[V/x] \\ (R\mu^\varepsilon): & \langle V \parallel \mu x^\ominus.c \rangle & \rhd_{\text{R}} & c[V/x] \\ (R1/\bot): & \langle () \parallel \mu().c \rangle & \rhd_{\text{R}} & c \\ (R\otimes/\parr): & \langle V \otimes W \parallel \mu(x \parr y).c \rangle & \rhd_{\text{R}} & c[V/x, W/y] \\ (R\oplus/\&): & \langle \iota_i(V) \parallel \mu[x_1.c_1 \mid x_2.c_2] \rangle & \rhd_{\text{R}} & c_i[V/x_i] \\ \text{(no rule } R\mathbf{0}/\top) & & & \end{array}$$

(b) Reduction rules

$$\begin{array}{rccl} (E\mu^+): & \mu x^+.\langle t \parallel x \rangle & \rhd_{\text{E}} & t \\ (E\mu^\ominus): & \mu x^\ominus.\langle x \parallel V \rangle & \rhd_{\text{E}} & V \\ (E1/\bot): & \mu().\langle () \parallel V \rangle & \rhd_{\text{E}} & V \\ (E\otimes/\parr): & \mu(x \parr y).\langle x \otimes y \parallel V \rangle & \rhd_{\text{E}} & V \\ (E\oplus/\&): & \mu\big[x.\langle \iota_1(x) \parallel V \rangle \big| y.\langle \iota_2(y) \parallel V \rangle\big] & \rhd_{\text{E}} & V \\ (E\mathbf{0}/\top)^{\lightning}: & \mu[x_1 \otimes \cdots \otimes x_n] & \rhd_{\text{E}} & V \end{array}$$

(c) Extensionality rules ($\lightning$: the η rule for $\mathbf{0}/\top$ is meaningful only under valid typing constraints)

Figure 2: $\mathbf{MALL}^\eta_p/\mathbf{LK}^\eta_p$: calculus

$$(D): \quad \big\langle t \,\big\|\, \mu x^\ominus.\langle u \parallel \mu y^\ominus.c \rangle \big\rangle \quad \rhd_{\text{D}} \quad \big\langle u \,\big\|\, \mu y^\ominus.\langle t \parallel \mu x^\ominus.c \rangle \big\rangle$$

(a) Depolarisation conversion

Figure 3: $\mathbf{MALL} = \mathbf{MALL}^\eta_p + \text{depolarisation}$

- $\mathbf{\Gamma, \Delta \ldots}$ are maps from a finite set of variables to types provided with total orders on their domain, notation $(x_1 : A_1, \ldots, x_n : A_n)$.
- Concatenation (Γ, Δ) is defined when the domains of Γ and Δ are disjoint.
- $\mathbf{LK}_p^\eta$: exchange, weakening and contractions on all formulae

$$\Sigma(\Gamma; \Gamma') = \left\{\sigma : \operatorname{dom}\Gamma \to \operatorname{dom}\Gamma' \,\middle|\, \Gamma'(\sigma(x)) = \Gamma(x)\right\}$$

- $\mathbf{MALL}_p^\eta$: exchange on all formulae, no contraction nor weakening

$$\Sigma^*(\Gamma; \Gamma') = \left\{\sigma \in \Sigma(\Gamma; \Gamma') \,\middle|\, \sigma \text{ bijective}\right\}$$

- Judgements are: $\quad c : (\vdash \Gamma) \qquad \vdash t : A \mid \Gamma$

(a) Judgements

$$\frac{}{\vdash x : A \mid x : \overline{A}}\ (\text{ax}) \qquad \frac{c : (\vdash x : A_\varepsilon, \Gamma)}{\vdash \mu x^\varepsilon.c : A_\varepsilon \mid \Gamma}\ (\mu^\varepsilon)$$

$$\frac{\vdash t : A \mid \Gamma \qquad \vdash u : \overline{A} \mid \Delta}{\langle t \parallel u \rangle : (\vdash \Gamma, \Delta)}\ (\text{cut})$$

(b) Identity

$$\frac{\vdash t : A \mid \Gamma}{\vdash t[\sigma] : A \mid \Gamma'}\ (\vdash \sigma) \qquad \frac{c : (\vdash \Gamma)}{c[\sigma] : (\vdash \Gamma')}\ (\sigma)$$

(c) Structure — $\sigma \in \Sigma^*(\Gamma; \Gamma')$ (for $\mathbf{MALL}_p^\eta$) or $\sigma \in \Sigma(\Gamma; \Gamma')$ (for $\mathbf{LK}_p^\eta$)

$$\frac{\vdash V : A \mid \Gamma \qquad \vdash W : B \mid \Delta}{\vdash V \otimes W : A \otimes B \mid \Gamma, \Delta}\ (\otimes^f) \qquad \frac{}{\vdash () : \mathbf{1} \mid}\ (\mathbf{1})$$

$$\frac{c : (\vdash x : A, y : B, \Gamma)}{\vdash \mu(x \parr y).c : A \parr B \mid \Gamma}\ (\parr) \qquad \frac{c : (\vdash \Gamma)}{\Gamma \vdash \mu().c : \bot \mid \Delta}\ (\bot)$$

$$\frac{c : (\vdash x : A, \Gamma) \qquad c' : (\vdash y : B, \Gamma)}{\vdash \mu[x.c | y.c'] : A \,\&\, B \mid \Gamma}\ (\&) \qquad \frac{\Gamma \vdash V : A \mid}{\Gamma \vdash \mu[V] : \top \mid}\ (\top^f)$$

$$\frac{\vdash V : A_i \mid \Gamma}{\vdash \iota_i(V) : A_1 \oplus A_2 \mid \Gamma}\ (\oplus_i^f) \qquad \text{(no rule for } \mathbf{0}\text{)}$$

(d) Logic

Figure 4: $\mathbf{MALL}_p^\eta/\mathbf{LK}_p^\eta$: simple types

$$\frac{\vdash t : A \mid \Gamma \qquad \vdash u : B \mid \Delta}{\vdash t \otimes u : A \otimes B \mid \Gamma, \Delta}\ (\vdash \otimes) \qquad t_\varepsilon \otimes u_{\varepsilon'} \stackrel{\text{def}}{=} \mu x^+.\left\langle t \,\middle\|\, \mu y^{\bar\varepsilon}.\left\langle u \,\middle\|\, \mu z^{\bar\varepsilon'}.\langle y \otimes z \parallel x \rangle\right\rangle\right\rangle$$

$$\frac{\Gamma \vdash t : (A_i)_\varepsilon \mid}{\Gamma \vdash \iota_i(t) : A_1 \oplus A_2 \mid}\ (\vdash \oplus_i) \qquad \iota_i(t_\varepsilon) \stackrel{\text{def}}{=} \mu x^+.\left\langle t \,\middle\|\, \mu y^{\bar\varepsilon}.\langle \iota_i(y) \parallel x \rangle\right\rangle$$

$$\frac{}{\Gamma \vdash \mu[]_\Gamma : \top \mid}\ (\vdash \top) \qquad \mu[]_\Gamma \stackrel{\text{def}}{=} \mu[x_1 \otimes \cdots \otimes x_n] \text{ pour } \{x_1, \ldots, x_n\} = \operatorname{dom}\Gamma$$

Figure 5: Derivable rules of $\mathbf{MALL}_p^\eta/\mathbf{LK}_p^\eta$

2 LL & LLNL

- Linear logic $\mathbf{LL}_p^\eta$: $\mathbf{MALL}_p^\eta$ + exponentials ?, !

$$\begin{array}{lrcl} \text{Positives:} & P, Q, A_+ & ::= & \ldots \mid\, !A \\ \text{Negatives:} & N, M, A_- & ::= & \ldots \mid\, ?A \end{array}$$

$$\overline{!A} \stackrel{\text{def}}{=} ?\overline{A} \quad \overline{?A} \stackrel{\text{def}}{=} !\overline{A}$$

Weakening and contractions for formulae of the form $?A$.

$$\Sigma^{\mathbf{LL}}(\Gamma;\Gamma') \stackrel{\text{def}}{=} \left\{\sigma \in \Sigma(\Gamma;\Gamma') \,\middle|\, \sigma \text{ bijective on formulae not of the form } ?A\right\}$$

- Linear/non-linear logic $\mathbf{LLNL}_p^\eta$: $\mathbf{LL}_p^\eta$ + structural rules extended to *?-algebras*:

$$\begin{array}{lrcl} \text{!-coalgebras} & P^!, Q^! & ::= & !A \mid \mathbf{1} \mid \mathbf{0} \mid P^! \otimes Q^! \mid P^! \oplus Q^! \\ \text{?-algebras} & N^?, M^? & ::= & ?A \mid \bot \mid \top \mid N^? \parr M^? \mid N^? \,\&\, M^? \end{array}$$

$$\Sigma^{\mathbf{LNL}}(\Gamma;\Gamma') \stackrel{\text{def}}{=} \left\{\sigma \in \Sigma(\Gamma;\Gamma') \,\middle|\, \sigma \text{ bijective on formulae not of the form } N^?\right\}$$

(a) Exponentials, algebras and structure maps

$$\text{(Co)Values:} \quad V, W \;::=\; \ldots \;\mid\; ?V \;\mid\; \mu!x.c$$

(b) Terms

$$(R!/?) : \quad \langle \mu!x.c \,\|\, ?V \rangle \;\rhd_{\mathrm{R}}\; c[V/x]$$

(c) Reduction rules

$$(E!/?) : \quad \mu!x.\langle V \,\|\, ?x \rangle \;\rhd_{\mathrm{E}}\; V$$

(d) Extensionality rules

$$\frac{\vdash t : A \mid \Gamma}{\vdash t[\sigma] : A \mid \Gamma'}\,(\vdash \sigma) \qquad \frac{c : (\vdash \Gamma)}{c[\sigma] : (\vdash \Gamma')}\,(\sigma)$$

(e) Structure for $\sigma \in \Sigma^{\mathbf{LL}}(\Gamma;\Gamma')$ or $\sigma \in \Sigma^{\mathbf{LNL}}(\Gamma;\Gamma')$.

$$\frac{c : (\vdash x : A, \Gamma^?)}{\vdash \mu!x.c : !A \mid \Gamma^?}\,(\text{prom.}) \qquad \frac{\vdash V : A \mid \Gamma}{\vdash ?V : ?A \mid \Gamma}\,(\text{der.}^f)$$

(f) Logic — where $(x_1 : A_1, \ldots, x_n : A_n)^?$ stands for the typing context $(x_1 : ?A_1, \ldots, x_n : ?A_n)$ for **LL** and $(x_1 : A_1^?, \ldots, x_n : A_n^?)$ for **LLNL**.

$$\frac{\vdash t : A \mid \Gamma}{\vdash ?t : ?A \mid \Gamma}\,(\text{der.}) \qquad ?t_\varepsilon \stackrel{\text{def}}{=} \mu x^\ominus.\big\langle \mu y^{\bar\varepsilon}.\langle x \,\|\, ?y \rangle \,\big\|\, t \big\rangle$$

(g) Remaining rule of **LL**/**LLNL** (without value restriction)

Figure 6: $\mathbf{LL}_p^\eta/\mathbf{LLNL}_p^\eta = \mathbf{MALL}_p^\eta$ + the above